Improper integrals:

e.g. $\int_0^{+\infty} f(x)\,dx$, $\int_{-\infty}^{-3} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$

$\int_1^3 \frac{1}{x-2}\,dx$, $\int_1^3 \frac{1}{(x-1)(x-4)}\,dx$, $\int_1^3 \frac{1}{(x-1)(x-3)}\,dx$

$$\underbrace{\int_0^{+\infty} e^x\,dx}_{} := \lim_{b\to+\infty} \int_0^b e^x\,dx = \lim_{b\to+\infty} (e^b - 1) = \underline{+\infty}.$$

diverges.

e^x is Riemann int. on any $\underbrace{[a,b]}_{\text{closed and bounded}} \subset [0,+\infty)$
(and bounded)

$$\underbrace{\int_0^{+\infty} e^{-x}\,dx}_{} = \lim_{b\to+\infty} \int_0^b e^{-x}\,dx = \lim_{b\to+\infty} (-e^{-b}+1) = 1.$$

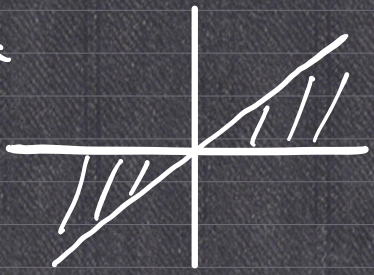
converges.

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$$

$$= \lim_{a\to-\infty} \int_a^0 f(x)\,dx + \lim_{b\to+\infty} \int_0^b f(x)\,dx$$

(!) $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to+\infty} \int_{-a}^{a} f(x)\,dx$ ✗

$$\int_{-\infty}^{\infty} x\,dx = \underbrace{\lim_{a\to-\infty}\int_a^0 x\,dx}_{} + \underbrace{\lim_{b\to+\infty}\int_0^b x\,dx}_{=+\infty.}$$



$$\lim_{a\to-\infty}\left[\frac{x^2}{2}\right]_a^0$$

$$= \lim_{a\to-\infty}\left(-\frac{a^2}{2}\right) = -\infty.$$

diverges.

$$\int_1^{+\infty}\frac{1}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^p}\,dx = \begin{cases} \lim_{b\to+\infty}(\log b - \log 1) = +\infty & \text{if } p=1 \\ \lim_{b\to\infty}\left(\dfrac{b^{1-p}-1}{1-p}\right) & \text{if } p\neq 1 \end{cases}$$

$$\lim_{b\to+\infty} b^{1-p} = 0 \quad \text{if} \quad 1-p<0 \iff p>1$$

$$\int_1^{+\infty}\frac{1}{x^p}\,dx = \frac{1}{p-1}$$

$$\lim_{b\to+\infty} b^{1-p} = +\infty \quad \text{if} \quad p<1$$

$\int_1^{+\infty}\frac{1}{x^p}\,dx$ diverges

Ex. $$\int_0^{+\infty} xe^{-x}\,dx = \lim_{b\to+\infty}\int_0^b xe^{-x}\,dx$$

$$= \lim_{b\to+\infty}\int_0^b x\,d(-e^{-x})$$

$$= \lim_{b\to+\infty}\left(\left[-xe^{-x}\right]_0^b + \int_0^b e^{-x}\,dx\right)$$

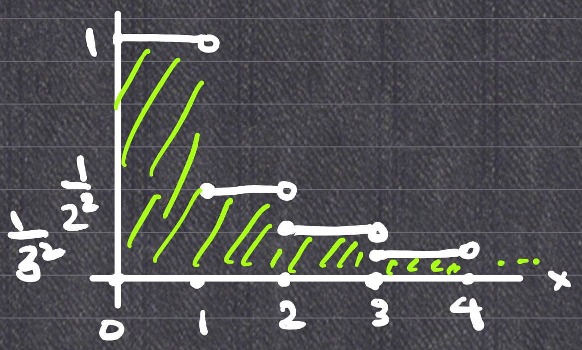
$$= \lim_{b\to+\infty}\left(\underbrace{-be^{-b}}_{\to 0 \text{ by L'Hospital}} + 1 - e^{-b}\right) = 1.$$

$$\int_0^{+\infty} xe^{-x}\,dx = \int_0^{+\infty} x\,d(-e^{-x}) = \underbrace{[-xe^{-x}]_0^{+\infty}}_{\lim\limits_{b\to+\infty}[-xe^{-x}]_0^{b}} + \underbrace{\int_0^{+\infty} e^{-x}\,dx}_{1}$$

Ex. 4.20

$$f(x) := \frac{1}{[x+1]^2}$$

$[x]$ = greatest integer $\leq x$.



Expect:

$$\int_0^{+\infty} f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\int_0^{+\infty} f(x)\,dx := \underbrace{\lim_{b\to+\infty}}_{\text{function limit}} \int_0^{b} f(x)\,dx$$

$$\lim_{n\to\infty} \int_0^{n} f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$$

↑
sequence limit

$$\int_0^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_0^{b} f(x)\,dx = \lim_{b\to+\infty}\left(\underbrace{\int_0^{[b]} f(x)\,dx}_{\downarrow\ \sum_{n=1}^{\infty}\frac{1}{n^2}} + \underbrace{\int_{[b]}^{b} f(x)\,dx}_{\downarrow\ 0} \right)$$

Claim: as $b \to +\infty$

$$\mathbb{R} \longrightarrow \mathbb{R}$$

$$b \mapsto \int_0^{[b]} f(x)\,dx$$

\to
\mapsto

$$\mathbb{R} \xrightarrow{\phi} \mathbb{N} \qquad \mathbb{N} \xrightarrow{\psi} \mathbb{R}$$

$$b \mapsto [b] \qquad N \mapsto \int_0^{N} f(x)\,dx$$

$$(\phi(b) = [b])$$

↑

$$\psi\circ\phi(b) = \psi([b]) = \int_0^{[b]} f(x)\,dx \qquad \psi(N) = \int_0^N f(x)\,dx$$

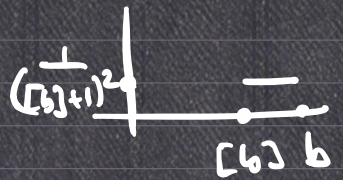
$$\left\{\begin{array}{l} \lim\limits_{b\to+\infty} \phi(b) = +\infty. \\ \lim\limits_{N\to+\infty} \psi(N) = \lim\limits_{N\to+\infty} \sum\limits_{n=1}^{N} \dfrac{1}{n^2} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}. \end{array}\right.$$

$$\lim_{b\to+\infty} \int_0^{[b]} f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{N\to\infty} \int_0^N f(x)\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\rightarrow \lim_{b\to+\infty} \psi\circ\phi(b) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \lim_{b\to+\infty} \int_0^{[b]} f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$$



$$0 \le \left| \int_{[b]}^{b} f(x)\,dx \right| \le \int_{[b]}^{b} |f(x)|\,dx$$

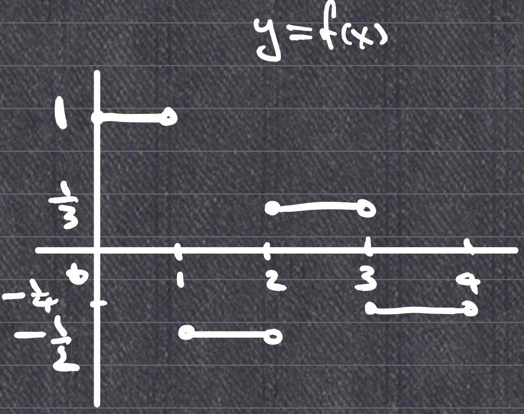
$$= \underbrace{(b-[b])}_{\le 1} \cdot \frac{1}{[b+1]^2} \le \frac{1}{[b+1]^2} < \frac{1}{b^2} \to 0.$$

as $b\to+\infty$.

$$b < [b+1] = [b]+1$$

Example of $f : [0,+\infty) \to \mathbb{R}$ such that

$$\left\{\begin{array}{ll} \int_0^{+\infty} |f(x)|\,dx & \text{diverges} \\ \int_0^{+\infty} f(x)\,dx & \text{converges.} \end{array}\right.$$

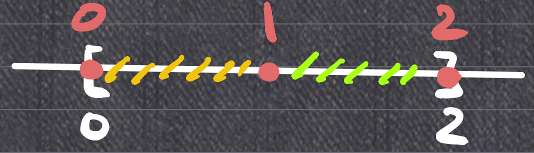


$$\int_0^{+\infty} |f(x)|\,dx = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty.$$

$$\int_0^{+\infty} f(x)\,dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots \quad \text{converges.}$$

$$f: [a,b] \to [-\infty, \infty]$$

$$f(x) = \frac{1}{x(x-1)(x-2)} \qquad \int_0^2 \frac{1}{x(x-1)(x-2)}\,dx = ?$$



$$\int_0^2 \frac{1}{x(x-1)(x-2)}\,dx := \underbrace{\int_0^1 \frac{dx}{x(x-1)(x-2)}} + \underbrace{\int_1^2 \frac{dx}{x(x-1)(x-2)}}$$

$$:= \underbrace{\lim_{a\to 0^+} \int_a^{1/2} \frac{1}{x(x-1)(x-2)}\,dx + \lim_{b\to 1^-} \int_{1/2}^{b} \frac{1}{x(x-1)(x-2)}\,dx}$$

$$+ \underbrace{\lim_{a\to 1^+} \int_a^{3/2} \frac{1}{x(x-1)(x-2)}\,dx + \lim_{b\to 2^-} \int_{3/2}^{b} \frac{1}{x(x-1)(x-2)}\,dx}$$

Keep in mind:

<u>$(p\neq 1)$</u> $$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \left[\frac{x^{-p+1}}{1-p}\right]_a^1$$

$$= \lim_{a\to 0^+} \frac{1-a^{-p+1}}{1-p} = \begin{cases} \frac{1}{1-p} & \text{if } -p+1>0 \\ +\infty & \text{if } -p+1<0 \end{cases}$$

$$\int_0^1 \frac{1}{x}\,dx = +\infty \qquad \int_0^1 \frac{1}{x^p}\,dx \text{ converges} \Leftrightarrow p<1$$

$$(\log x)' = \frac{1}{x}.$$

$$\lim_{a\to 0^+} \int_a^{1/2} \frac{1}{x(x-1)(x-2)}\,dx$$

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

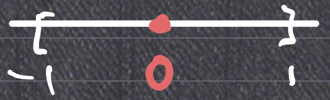
$$= \frac{1}{2x} - \frac{1}{x-1} + \frac{1}{2(x-2)}$$

$$= \lim_{a \to 0^+} \int_a^{1/2} \left(\frac{1}{2x} - \frac{1}{x-1} + \frac{1}{2(x-2)} \right) dx$$

diverges $\quad \int_0^{1/2} \frac{1}{2x}\,dx = +\infty.$

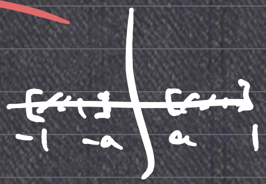
$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

diverges.



$\therefore$ diverges

$$\cancel{\int_{-1}^{1} \frac{1}{x}\,dx = \lim_{a \to 0} \left(\int_{-1}^{-a} \frac{1}{x}\,dx + \int_{a}^{1} \frac{1}{x}\,dx \right)}$$



$$\int_0^{+\infty} f(x)\,dx := \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$$

when $f(x) \to +\infty$ as $x \to 0^+$

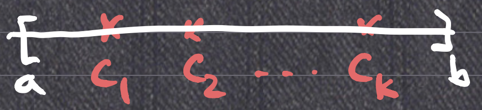
f is locally Riemann integrable on $(0, +\infty)$.

e.g.

$$\int_0^{+\infty} \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx$$

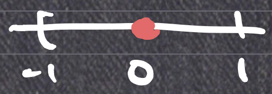
✓ X

diverges.



$$\int_a^b f(x)\,dx := \underbrace{\int_a^{c_1} f(x)\,dx}_{\lim\limits_{\beta\to c_1^-}\int_a^{\beta} f(x)\,dx} + \underbrace{\int_{c_1}^{c_2} f(x)\,dx}_{\int_{c_1}^{\frac{c_1+c_2}{2}} f(x)\,dx + \int_{\frac{c_1+c_2}{2}}^{c_2} f(x)\,dx} + \dots + \int_{c_k}^{b} f(x)\,dx$$

$$\int_{-1}^{1} \frac{1}{x}\,dx \underbrace{:=}_{\text{by definition}} \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$



$$= \underbrace{\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x}\,dx}_{\lim\limits_{a\to 0^+} \int_{-1}^{-a} \frac{1}{x}\,dx} + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\,dx$$

$$\lim_{a\to 0^+}\left(\int_{-1}^{-a} \frac{1}{x}\,dx + \int_{a}^{1} \frac{1}{x}\,dx \right) \neq \lim_{a\to 0^+} \int_{-1}^{-a} \frac{1}{x}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\,dx$$