香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

MIDTERM EXAMINATION

Course Code:	MATH 1024
Course Title:	Honors Calculus II
Semester:	Spring 2017-18
Date and Time:	10:00AM-1:00PM, 24 March 2018

Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- It is an **OPEN-NOTES** exam. Authorized reference materials are the lecture notes, two of your notebooks, and **your own** homework. No other reference materials are allowed.
- Answer **ALL** problems. Write your solutions in the space provided.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ______________________________

Student's Name (English): ______________________________
FAMILY NAME, First Name

HKUST ID: ____________________ **Seat Number:** ____________

Part A - Short Questions (25 points)

[Recommended time: $<$ 30 min.]

1. Which ONE of the following is the correct definition of the lower Riemann integral **[3]**

$$L\int_a^b f(x)\,dx, \quad \text{or equivalently} \quad \underline{\int_a^b} f(x)\,dx$$

for a bounded function $f(x)$ over a bounded interval $[a,b]$? Put ✓ in the correct answer:

◯ $\sup\left\{\sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1}) \,\middle|\, P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}\right\}$

◯ $\sup\left\{\sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1}) \,\middle|\, P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}\right\}$

◯ $\inf\left\{\sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1}) \,\middle|\, P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}\right\}$

◯ $\inf\left\{\sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1}) \,\middle|\, P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}\right\}$

2. Let $f(x)$ be a third-degree polynomial such that $f(x) > 0$, $f'(x) > 0$ and $f''(x) < 0$ on $[0,1]$. Denote P to be the uniform partition of $[0,1]$ into 100 sub-intervals. Rank the following sums and integrals in increasing order using $<$ or $=$. Do NOT use $\leq$. No justification is needed. **[5]**

$$U(P,f) \qquad L(P,f) \qquad L_{100} \qquad M_{100} \qquad R_{100} \qquad T_{100} \qquad S_{100} \qquad \int_0^1 f(x)\,dx$$

3. Which of the following statement(s) would imply a (non-empty) set A in $\mathbb{R}^2$ has zero Jordan measure? Put ✓ in ALL correct answer(s), or put ✓ in "None of the above". **[4]**

◯ $\mu_*(A) = 0$

◯ $\mu^*(A) = 0$

◯ There is a simple region S such that $S \subset A$ and $\mu(S) = 0$.

◯ There is a simple region S such that $A \subset S$ and $\mu(S) = 0$.

◯ None of the above

4. Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ such that $f(x)$ itself is not Riemann integrable on $[0,1]$, but $f(x)^2$ is Riemann integrable on $[0,1]$. **[3]**

5. Show that π^x is Riemann integrable on $[0,1]$ from the definition, and find the value of the integral: **[5]**

$$\int_0^1 \pi^x\,dx.$$

6. Consider the function: **[5]**

$$f(x) = \frac{1}{(x-1)(x-2)\cdots(x-1024)}.$$

Determine all real numbers c such that $\int_c^{+\infty} f(x)\,dx$ converges. Explain your answer.

Part B - Long Questions (75 points): Answer ALL THREE problems

[Recommended time: Q1 < 40m, Q2 < 55m, Q3 < 55m.]

1. Consider a bijective function $f : [a, b] \to [f(a), f(b)]$ which is differentiable on $[a, b]$ and $f'(x) > 0$ on (a, b).
   (a) By sketching a diagram, guess the value of: **[4]**
   $$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$
   in terms of a, b, $f(a)$, $f(b)$.
   (b) Prove your claim in (a) using integration by substitution. **[6]**
   (c) Let $g(x) = 5\sqrt{x} - 6$. Show that the definite integral: **[10]**
   $$\int_4^9 g(g(g(g(g(x)))))\,dx$$
   is a rational number.

2. Let $f(x) : [a, b] \to \mathbb{R}$ be a continuous function defined on a bounded interval $[a, b]$. Fix a constant $\lambda \in [0, 1]$ and a large positive integer n. Consider the **uniform** partition of $[a, b]$:
   $$\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$
   and define a numerical approximation of $\int_a^b f(x)\,dx$ by:
   $$A_n := \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1}), \quad \text{where } x_i^* := (1-\lambda)x_{i-1} + \lambda x_i.$$
   (a) By sketching a diagram, explain the geometric meaning of the sum A_n. **[4]**
   (b) Suppose $f'(x)$ exists and is continuous on $[a, b]$. Show that: **[10]**
   $$\left|\int_a^b f(x)\,dx - A_n\right| \leq \frac{(1-2\lambda+2\lambda^2)(b-a)^2}{2n} \sup_{[a,b]} |f'|.$$
   (c) Suppose further that $f''(x)$ exists and is continuous on $[a, b]$. Show that: **[15]**
   $$\left|\int_a^b f(x)\,dx - A_n\right| \leq \frac{|1-2\lambda|\,(b-a)^2}{2n} \sup_{[a,b]} |f'| + \frac{(1-3\lambda+3\lambda^2)(b-a)^3}{6n^2} \sup_{[a,b]} |f''|.$$

3. Let a and b be two positive integers, and $g_0 : [0, \frac{a}{b}] \to \mathbb{R}$ be a continuous function on $[0, \frac{a}{b}]$, we define for any integer $n \geq 0$ that:
   $$f_n(x) := \frac{x^n(a-bx)^n}{n!}$$
   $$g_n(x) := \int_0^x g_{n-1}(x)\,dx \qquad \text{for any } n \geq 1$$
   (a) Show that for any $n \geq 0$, the k-th order derivatives $f_n^{(k)}(0)$ and $f_n^{(k)}(\frac{a}{b})$ are integers for any $k \geq 0$. **[6]**

(b) Define $I_n := \int_0^{\frac{a}{b}} f_n(x) g_0(x)\, dx$. Using (a), or otherwise, show that there exist two **finite** sequences of integers $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ (a_i, b_i and N may depend on n) such that **[10]**

$$I_n = \sum_{i=1}^{N} (a_i g_i(a/b) + b_i g_i(0))$$

(c) Given that r is a positive number such that there exists a **non-negative**, **non-constant** continuous function $g_0 : [0, r] \to [0, \infty)$ such that $g_n(0)$ and $g_n(r)$ are integers for any $n \geq 0$ (where $g_n(x)$ is defined recursively as above). Show that r is irrational. **[10]**