4.7 Improper Integrals

An integral $\int_a^b f(x)\,dx$ is called an **improper integral** if one of $a=-\infty$ or $b=+\infty$, and/or f is unbounded on $[a,b]$.

4.7.1 Integral over an unbounded interval

Let's first discuss the case when the integral is defined over an unbounded interval, but f is *locally Riemann integrable* on the domain of f, meaning that for any closed and bounded interval $[a,b] \subset \text{domain}(f)$, the function f is **bounded** and **Riemann integrable** on $[a,b]$. Here the bound can depend on a and b. One example is $f(x)=e^x$. It is not bounded on $\mathbb{R}$, but is bounded (by e^b) on each closed and bounded interval $[a,b]$. Here are the definitions of such improper integrals:

$$\begin{aligned}
\int_0^{+\infty} f(x)\,dx &:= \lim_{b\to+\infty}\int_0^b f(x)\,dx \\
\int_{-\infty}^0 f(x)\,dx &:= \lim_{a\to-\infty}\int_a^0 f(x)\,dx \\
\int_{-\infty}^{+\infty} f(x)\,dx &:= \int_0^{+\infty} f(x)\,dx + \int_{-\infty}^0 f(x)\,dx \\
&= \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx
\end{aligned}$$

Similar to an infinite series, we say $\int_0^{+\infty} f(x)\,dx$ converges if the limit $\lim_{b\to+\infty}\int_0^b f(x)\,dx$ exists, and say $\int_0^{+\infty} f(x)\,dx$ diverges if it does not exist (including the case when the limit is infinity).

ⓘ The value 0 in the third integral can be generally replaced by any other constant:

$$\int_{-\infty}^{+\infty} = \int_c^{+\infty} + \int_{-\infty}^c .$$

One can show that if f is Riemann integrable on every closed and bounded interval $[a,b]$, then we have

$$\lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to+\infty}\int_c^b f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx$$

for any $c\in\mathbb{R}$.

■ **Example 4.16** Let $p\in\mathbb{R}$, then

$$\int_1^b \frac{1}{x^p}\,dx = \begin{cases} \left[\dfrac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=b} = \dfrac{b^{1-p}-1}{1-p} & \text{if } p\neq 1 \\ \log b & \text{if } p=1 \end{cases}$$

As $b\to+\infty$, we know $b^{1-p}\to 0$ if $p>1$, and $b^{1-p}=+\infty$ if $p<1$. Also, $\log b\to+\infty$ too. Combining all these, we conclude that

$$\int_1^{+\infty}\frac{1}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p>1 \\ +\infty & \text{if } p\leq 1 \end{cases}.$$

We may say, for instance, $\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx$ diverges.

For an improper integral such as $\int_{-\infty}^{+\infty} f(x)\,dx$, we need both $\int_0^{+\infty} f(x)\,dx$ and $\int_{-\infty}^{0} f(x)\,dx$ converge:

■ Example 4.17

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to+\infty}\int_0^b \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 + \lim_{b\to+\infty}\left[\tan^{-1}x\right]_0^b \\
&= \lim_{a\to-\infty}(-\tan^{-1}a) + \lim_{b\to+\infty}\tan^{-1}b \\
&= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.
\end{aligned}$$

However, for $\int_{-\infty}^{+\infty} e^{-x}\,dx$, we see that:

$$\int_{-\infty}^{0} e^{-x}\,dx = \lim_{a\to-\infty}(-e^0 + e^{-a}) = +\infty.$$

Therefore, $\int_{-\infty}^{+\infty} e^{-x}\,dx$ diverges even though $\int_0^{+\infty} e^{-x}\,dx$ converges.

Note that

$$\int_{-\infty}^{+\infty} x\,dx := \lim_{a\to-\infty}\int_a^0 x\,dx + \lim_{b\to+\infty}\int_0^b x\,dx$$

but NOT:

$$\lim_{a\to+\infty}\int_{-a}^{a} x\,dx,$$

which is one misconception for many non-math majors (and some math majors too).

For improper integrals involving the use of integration by parts or by substitution, one can just do it as the proper case before taking limits.

■ Example 4.18

$$\begin{aligned}
\int_0^{+\infty} xe^{-x}\,dx &= \lim_{b\to+\infty}\int_0^b xe^{-x}\,dx \\
&= \lim_{b\to+\infty}\left(\int_0^b x d(-e^{-x})\right) \\
&= \lim_{b\to+\infty}\left([-xe^{-x}]_0^b - \int_0^b (-e^{-x})\,dx\right) \\
&= \lim_{b\to+\infty}\left(-be^{-b} - e^{-b} + 1\right) = 0 + 0 + 1 = 1.
\end{aligned}$$

Here we have used L'Hospital's rule to compute the limit of be^{-b} as $b \to +\infty$.

■ **Example 4.19**

$$\begin{aligned}\int_2^{+\infty} \frac{1}{x(\log x)^2}\, dx &= \lim_{b\to+\infty} \int_2^b \frac{1}{x(\log x)^2}\, dx \\ &= \lim_{b\to+\infty} \int_{x=2}^{x=b} \frac{1}{(\log x)^2}\, d(\log x) \\ &= \lim_{b\to+\infty} \left[-\frac{1}{\log x}\right]_{x=2}^{x=b} \\ &= \lim_{b\to+\infty} \left(-\frac{1}{\log b} + \frac{1}{\log 2}\right) = \frac{1}{\log 2}.\end{aligned}$$

■ **Example 4.20** Let $f : [0,\infty) \to \mathbb{R}$ the function

$$f(x) = \frac{1}{[x+1]^2}.$$

Then, $f(x) = 1$ on $[0,1)$, $f(x) = \frac{1}{2^2}$ on $[1,2)$, and $f(x) = \frac{1}{3^2}$ on $[2,3)$ etc. By sketching the graph, it is easy to expect that $\displaystyle\int_0^{+\infty} f(x)\, dx$ should be equal to the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (which is $\frac{\pi^2}{6}$). Let's verify that it is really the case:

Note that $[a] \le a$ for any $a \in [0,\infty)$, we have

$$\int_0^a f(x)\, dx = \int_0^{[a]} f(x)\, dx + \int_{[a]}^a f(x)\, dx.$$

Note that on the interval $[0,[a]]$, the region $G^+_{[0,[a]]}(f)$ under the graph of f is a simple region, so by definition we have

$$\int_0^{[a]} f(x)\, dx = 1 + \frac{1}{2^2} + \cdots + \frac{1}{[a]^2} = \sum_{n=1}^{[a]} \frac{1}{n^2}$$

By the composition rule applied on the maps

$$\begin{aligned} \mathbb{R} &\to \mathbb{N} & \qquad \mathbb{N} &\to \mathbb{R} \\ a &\mapsto [a] & \qquad N &\mapsto \sum_{n=1}^{N} \frac{1}{n^2} \end{aligned}$$

one can then show, by taking composition of the above maps, we have

$$\lim_{a\to+\infty} \int_0^{[a]} f(x)\, dx = \lim_{a\to+\infty} \sum_{n=1}^{[a]} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For the remaining integral over $[[a],a]$, we consider:

$$\left|\int_{[a]}^a f(x)\, dx\right| \le \int_{[a]}^a |f(x)|\, dx \le (a-[a]) \sup_{[[a],a]} |f| \le 1 \cdot \frac{1}{([a]+1)^2} \le \frac{1}{a^2}.$$

By squeeze theorem, we conclude that $\lim_{a\to+\infty}\int_{[a]}^{a} f(x)\,dx = 0$. Hence, we proved

$$\int_0^{+\infty} f(x)\,dx = \lim_{a\to 0}\int_0^{[a]} f(x)\,dx + \lim_{a\to+\infty}\int_{[a]}^{a} f(x)\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

■ **Exercise 4.65** Find the value of the following improper integrals, or show that it diverges.

(a) $\displaystyle\int_{-\infty}^{0} e^x\,dx$

(b) $\displaystyle\int_{0}^{+\infty} \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx$

(c) $\displaystyle\int_{1}^{+\infty} \frac{\log x}{x^2}\,dx$

■ **Exercise 4.66** Suppose $\sum_{n=1}^{\infty} a_n$ converges to L. Consider the function $f : [0,\infty)\to\mathbb{R}$ where $f(x) = a_{[x]+1}$ for any $x\in[0,\infty)$. Show that

$$\int_0^{+\infty} f(x)\,dx = L.$$

Show also that if $\sum_{n=1}^{\infty} a_n$ diverges (as an infinite series), then $\displaystyle\int_0^{+\infty} f(x)\,dx$ also diverges (as an improper integral).

■ **Exercise 4.67** Construct a function $f : [0,\infty)\to\mathbb{R}$ such that $\displaystyle\int_0^{+\infty} f(x)\,dx$ converges, but $\displaystyle\int_0^{+\infty} |f(x)|\,dx$ diverges. [Hint: Use the previous exercise.]

4.7.2 Integral of an unbounded function

Next we consider functions which are unbounded whereas its domain could be bounded or unbounded. For Riemann integrals, we focus on functions of the following class:

$$\mathcal{F} := \{f : [a,b]\to[-\infty,\infty] : f \text{ is locally Riemann integrable on } [a,b]\backslash\{\text{finite set of points}\}\}.$$

Functions beyond this class are better handled using Lebesgue integrals, since the definition of Riemann integrals rely heavily on the use of intervals and sub-intervals.

For any function f which tends to $\pm\infty$ when $x\to a^+$ and $x\to b^-$, but is locally Riemann integrable on (a,b), we define its integral by

$$\int_a^b f(x)\,dx := \lim_{\alpha\to a^-}\int_\alpha^c f(x)\,dx + \lim_{\beta\to b^+}\int_c^\beta f(x)\,dx$$

where c is any constant in (a,b). If f is locally Riemann integrable on $[a,b)$ and tends $\pm\infty$ when $x\to b^+$, then we define

$$\int_a^b f(x)\,dx := \lim_{\beta\to b^+}\int_a^\beta f(x)\,dx,$$

and similarly for a function f which is locally Riemann integrable on $(a,b]$ and tends to $\pm$ when $x\to a$.

Now for a general function $f\in\mathcal{F}$, we first locate all points $c_1 < c_2 < \cdots < c_k$ in (a,b) at which f tends to $\pm\infty$, then we define

$$\int_a^b f(x)\,dx := \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{k-1}}^{c_k} f(x)\,dx + \int_{c_k}^{b} f(x)\,dx.$$

If just one of the above integrals diverges, we say $\int_a^b f(x)\,dx$ diverges (even if just one of them diverges)

■ **Example 4.21**

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}[2\sqrt{x}]_a^1 \\ &= \lim_{a\to 0^+}(2-2\sqrt{a}) = 2\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{1}{x(1-x)}\,dx &= \int_0^1 \left(\frac{1}{x}+\frac{1}{1-x}\right)dx \\ &= \lim_{a\to 0^+}\int_a^{1/2}\left(\frac{1}{x}+\frac{1}{1-x}\right)dx + \lim_{b\to 1^-}\int_{1/2}^b \left(\frac{1}{x}+\frac{1}{1-x}\right)dx \\ &= \lim_{a\to 0^+}[\log(x)+\log(1-x)]_a^{1/2} + \lim_{b\to 1^-}\int_{1/2}^b \left(\frac{1}{x}+\frac{1}{1-x}\right)dx.\end{aligned}$$

Note that

$$\lim_{a\to 0^+}[\log(x)+\log(1-x)]_a^{1/2} = \lim_{a\to 0^+}(\log(1/2)-\log a+\log(1/2)-\log(1-a))$$

does not exists (since $\log a \to -\infty$). We conclude that $\int_0^1 \frac{1}{x(1-x)}\,dx$ diverges. Note that there is no need to discuss the convergence of $\lim_{b\to 1^-}\int_{1/2}^b \left(\frac{1}{x}+\frac{1}{1-x}\right)dx$, as whether or not it converges the integral $\int_0^1 \frac{1}{x(1-x)}\,dx$ would still diverge.

■ **Example 4.22** Consider the improper integral

$$\int_0^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx.$$

The integral is *improper* in two ways: it is unbounded near 0, 1 and 2; and the interval $[0,\infty)$ of integration is unbounded. Therefore, we should first express:

$$\begin{aligned}&\int_0^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx \\ &= \int_0^1 \frac{1}{x(x-1)(x-2)}\,dx + \int_1^2 \frac{1}{x(x-1)(x-2)}\,dx \\ &\quad \int_2^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx.\end{aligned}$$

In each of the three integrals on the RHS, both the lower and upper end-points are “bad” points, so we need to further consider:

$$\int_0^1 \frac{1}{x(x-1)(x-2)}\,dx = \lim_{a\to 0^+}\int_a^{1/2} \frac{1}{x(x-1)(x-2)}\,dx + \lim_{b\to 1^-}\int_{1/2}^b \frac{1}{x(x-1)(x-2)}\,dx$$

$$\int_1^2 \frac{1}{x(x-1)(x-2)}\,dx = \lim_{a\to 1^+}\int_a^{3/2} \frac{1}{x(x-1)(x-2)}\,dx + \lim_{b\to 2^-}\int_{3/2}^b \frac{1}{x(x-1)(x-2)}\,dx$$

$$\int_2^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx = \lim_{a\to 0^+}\int_a^{3} \frac{1}{x(x-1)(x-2)}\,dx + \lim_{b\to +\infty}\int_{3}^b \frac{1}{x(x-1)(x-2)}\,dx$$

If one of the above six integrals diverges, it suffices to claim that $\int_0^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx$ diverges. In fact by observing that

$$\frac{1}{x(x-1)(x-2)} = \frac{1}{2x} - \frac{1}{x-1} + \frac{1}{2(x-2)},$$

one can already conclude that

$$\lim_{a\to 0^+}\int_a^{1/2} \frac{1}{x(x-1)(x-2)}\,dx$$

does not exist as $\int_0^{1/2} \frac{1}{2x}\,dx$ diverges, and hence $\int_0^{+\infty} \frac{1}{x(x-1)(x-2)}\,dx$ diverges.

■ **Example 4.23** It is good to keep in mind that

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^p}\,dx = \begin{cases} \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right) = \frac{1}{1-p} & \text{if } p<1 \\ \lim_{a\to 0^+}[\log x]_a^1 = +\infty & \text{if } p=1 \\ \lim_{a\to 0^+}\left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right) = +\infty & \text{if } p>1 \end{cases}$$

It is in contrast to $\int_1^{+\infty} \frac{1}{x^p}\,dx$, which converges when $p>1$.

■ **Example 4.24** The function $\frac{1}{x^2}$ is unbounded as $x\to 0$, and is locally Riemann integrable on $[-1,1]\backslash\{0\}$, so:

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx.$$

By the above example, we know that $\int_0^1 \frac{1}{x^2}\,dx$ diverges, so $\int_{-1}^1 \frac{1}{x^2}\,dx$ diverges too. It is *improper* to compute the *improper* integral this way:

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^1 = -2 \qquad \text{(WRONG!)}$$

Furthermore, it is *improper* to compute the integral like this:

$$\int_{-1}^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left(\int_{-1}^{-a} \frac{1}{x}\,dx + \int_a^1 \frac{1}{x}\,dx\right) \qquad \text{(WRONG!)},$$

but instead we should follow the definition:

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x},\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\,dx$$

which diverges by the previous example.

■ **Exercise 4.68** Determine whether the following improper integrals converge. If so, find its value.

(a) $\displaystyle\int_0^{+\infty} \frac{\log x}{x^2}\,dx$

(b) $\displaystyle\int_0^{1} \frac{\log x}{\sqrt{x}}\,dx$

(c) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

(d) $\displaystyle\int_{-2}^{2} \frac{1}{1-x^2}\,dx$

4.7.3 Comparison Tests

Very often, we are not concerned about the exact value of an improper integral, but we care about whether it converges or not. In number theory and complex analysis, we study an important function known as the Gamma function:

$$\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\,dt.$$

Although it can be shown (see Exercise 4.70) that $\Gamma(n) = (n-1)!$ for any $n \in \mathbb{N}$, but other values of $\Gamma(x)$ cannot be easily found. However, we can prove it converges when $x > 0$ by comparing $\Gamma(x)$ with another computable integral (note that $t^{x-1}e^{-t} \le e^{-t/2}$ when t is sufficiently large).

Proposition 4.27 — Comparison Test for Improper Integrals. Suppose f, g are locally Riemann integrable on $[a, +\infty)$ and $0 \le f(x) \le g(x)$ on $[N, +\infty)$ for sufficiently large $N \ge a$, then

- If $\displaystyle\int_a^{+\infty} g(x)\,dx$ converges, then $\displaystyle\int_a^{+\infty} f(x)\,dx$ converges.
- If $\displaystyle\int_a^{+\infty} f(x)\,dx$ diverges, then $\displaystyle\int_a^{+\infty} g(x)\,dx$ diverges.

Similar comparison tests hold for other types of improper integrals.

Proof. The second result is the contrapositive of the first one, so it suffices to prove the first one only. Define

$$F(x) := \int_a^x f(t)\,dt \quad \text{and} \quad G(x) := \int_a^x g(t)\,dt.$$

By $0 \le f(x) \le g(x)$ on $[N, +\infty)$, we know that $F(x)$ and $G(x)$ are both monotonically increasing on $[a, +\infty)$, and $F(x) \le G(x)$ on $[N, +\infty)$. Note that $\displaystyle\int_a^{+\infty} g(t)\,dt$ being convergent means the limit $\displaystyle\lim_{x\to+\infty} G(x)$ exists, so $G(x)$ is bounded on $[N, +\infty)$.

This shows $F(x)$ is also bounded on $[N, +\infty)$. Combining with the fact that $F(x)$ is monotone, we conclude that $\displaystyle\lim_{x\to+\infty} F(x)$ exists, and equivalently $\displaystyle\int_N^{+\infty} f(x)\,dx$ (and hence $\displaystyle\int_a^{+\infty} f(x)\,dx$) converges. ■

■ **Example 4.25** On $[2,\infty)$, we have

$$0 \leq \frac{1}{x^{2/3}} \leq \frac{1}{\sqrt{x^2-1}}.$$

Note that $\int_2^{+\infty} \frac{1}{x^{2/3}}\,dx$ diverges, so $\int_2^{+\infty} \frac{1}{\sqrt[3]{x^2-1}}\,dx$ diverges too. On the other hand, on $(0,\frac{1}{2}]$,

$$0 \leq \frac{1}{\sqrt[3]{x^2-1}} \leq \frac{1}{\sqrt[3]{\frac{1}{2}x^2}} = \frac{\sqrt[3]{2}}{x^{2/3}}.$$

Since $\frac{2}{3} < 1$, we know that

$$\int_0^{1/2} \frac{\sqrt[3]{2}}{x^{2/3}}\,dx$$

converges, so $\int_0^{1/2} \frac{1}{x^{2/3}}\,dx$ converges too.

The condition $0 \leq f(x)$ is necessary in Proposition 4.27. Here is a counterexample:

$$-\frac{1}{x} \leq \frac{1}{x^2} \quad \text{on } [1,\infty)$$

but $-\int_1^{+\infty} \frac{1}{x}\,dx$ diverges while $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges.

Sometimes it may be tricky to handle non-essential terms (such as the -1 in $\sqrt{x^2-1}$) when using the comparison test. The limit comparison test below lets us focus on the most important term when doing comparisons:

Proposition 4.28 — Limit Comparison Test for Improper Integrals. Suppose f,g are locally Riemann integrable on $[a,+\infty)$, and $f(x),g(x) \geq 0$ on $[a,+\infty)$. Consider the limit

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} =: L.$$

Then, we have

- If $L \in (0,+\infty)$, then $\int_a^{+\infty} f(x)\,dx$ if and only if $\int_a^{+\infty} g(x)\,dx$ converges.
- If $L = 0$, then $\int_a^{+\infty} g(x)\,dx$ converges implies $\int_a^{+\infty} f(x)\,dx$ converges.
- If $L = +\infty$, then $\int_a^{+\infty} g(x)\,dx$ converges implies $\int_a^{+\infty} f(x)\,dx$ converges.

Similar results hold for other types of improper integrals.

Proof. The proof simply follows from Proposition 4.27 and the order rule. When $0 < L < +\infty$, then for sufficiently large x, we have

$$\frac{L}{2} \leq \frac{f(x)}{g(x)} \leq 2L \implies \frac{L}{2}g(x) \leq f(x) \leq 2Lg(x).$$

If $L = 0$, then $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0$ shows $\frac{f(x)}{g(x)} \leq 1$ for sufficiently large x, and hence $f(x) \leq g(x)$. Applying Proposition 4.27 yields the desired result.

The case $L = +\infty$ follows from swapping f and g in the $L = 0$ case. ■

■ **Example 4.26** Consider
$$\int_3^{+\infty} \frac{|\sin x|}{(x-1)(x-2)}\,dx.$$
First note that 1 and 2 are outside the interval $[3,+\infty)$, so they are not considered as "bad points". Noting that on $[3,+\infty)$ we have:
$$0 \le \frac{|\sin x|}{(x-1)(x-2)} \le \frac{1}{(x-1)(x-2)}.$$
We next argue that $\displaystyle\int_3^{+\infty} \frac{1}{(x-1)(x-2)}\,dx$ converges. Considering that
$$\lim_{x\to+\infty} \frac{\frac{1}{(x-1)(x-2)}}{\frac{1}{x^2}} = 1$$
and $\displaystyle\int_3^{+\infty} \frac{1}{x^2}\,dx$ converges, the limit comparison test shows $\displaystyle\int_3^{+\infty} \frac{1}{(x-1)(x-2)}\,dx$ converges, and by comparison test,
$$\int_3^{+\infty} \frac{|\sin x|}{(x-1)(x-2)}\,dx$$
converges too.

■ **Example 4.27** Consider
$$\int_0^5 \frac{1}{\sqrt[3]{5x+2x^4}}\,dx.$$
Note that 0 is the "bad point" for this improper integral. As $x \to 0$, the term $5x$ is much larger than that of $2x^4$, so we expect $\sqrt[3]{5x+2x^4}$ behaves like $\sqrt[3]{x}$ as $x \to 0$. It suggests that we should compare it with
$$\lim_{x\to 0^+} \frac{\frac{1}{\sqrt[3]{5x+2x^4}}}{\frac{1}{\sqrt[3]{x}}} = \lim_{x\to 0^+} \sqrt[3]{\frac{x}{5x+2x^4}} = \lim_{x\to 0^+} \sqrt[3]{\frac{1}{5+2x^3}} = \frac{1}{\sqrt[3]{5}} \in (0,\infty).$$
Recall that $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{x}}\,dx$ converges, so by limit comparison test, we conclude that $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5x+2x^4}}\,dx$ converges too.

■ **Example 4.28** Consider the integral:
$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$
In order to show that it converges, we need to show both $\displaystyle\int_{-\infty}^{0} e^{-x^2}\,dx$ and $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx$ converge.

On $[1,+\infty)$, we have $x^2 \ge x$ and so $0 \le e^{-x^2} \le e^{-x}$. Since $\displaystyle\int_0^{+\infty} e^{-x}\,dx$ converges (to 1), by comparison test, we know that $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx$ converges.

For $\displaystyle\int_{-\infty}^{0} e^{-x^2}\,dx$, one can use the change of variables $y = -x$ to show that in fact

$\int_{-\infty}^{0} e^{-x^2}\,dx = \int_{0}^{+\infty} e^{-x^2}\,dx$. Alternatively, one can compare it with $e^{-x^2} \leq e^x$ on $(-\infty, -1]$. This concludes that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{+\infty} e^{-x^2}\,dx$$

converges.

■ **Exercise 4.69** Determine whether the following improper integrals converge or not.

(a) $\displaystyle\int_{2}^{+\infty} \frac{x\tan^{-1}x}{\sqrt{x^5 - 2x^2 + 1}}\,dx$

(b) $\displaystyle\int_{0}^{1} \frac{x\tan^{-1}x}{\sqrt{x^5 - 2x^2 + 1}}\,dx$

(c) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{x + \sqrt{x + \sqrt{x}}}}\,dx$

(d) $\displaystyle\int_{1}^{+\infty} \frac{1}{\sqrt{x + \sqrt{x + \sqrt{x}}}}\,dx$

■ **Exercise 4.70 — Gamma Function.** Let

$$\Gamma(x) := \int_{0}^{+\infty} t^{x-1} e^{-t}\,dt.$$

(a) Show that for any $x > 0$, the defining integral of $\Gamma(x)$ converges.

(b) Show that $\Gamma(x+1) = x\Gamma(x)$ for any $x > 0$, and deduce that $\Gamma(n) = (n-1)!$ for any $n \in \mathbb{N}$.

■ **Exercise 4.71 — Source: MATH1024 Spring 2018 Midterm.** Consider the function

$$f(x) = \frac{1}{(x-1)(x-2)\cdots(x-1024)}.$$

Determine all real numbers c such that $\int_{c}^{+\infty} f(x)\,dx$ converges. Explain your answer.

4.7.4 Absolute convergence versus conditional convergence

An improper integral such as $\int_{a}^{+\infty} f(x)\,dx$ is said to *converge absolutely* if $\int_{a}^{+\infty} |f(x)|\,dx$ converges. By the proposition below, one can also claim that $\int_{a}^{+\infty} f(x)\,dx$ converges too:

Proposition 4.29 — Absolute Convergence Test for Improper Integrals. Suppose f is locally Riemann integrable on $[a, \infty)$, and that $\int_{a}^{+\infty} |f(x)|\,dx$ converges, then $\int_{a}^{+\infty} f(x)\,dx$ converges too. Similar results hold for other types of improper integrals.

Proof. The key observation is $0 \leq f(x) + |f(x)| \leq 2\,|f(x)|$. By comparison test, we get that

$$\int_{a}^{+\infty} (f(x) + |f(x)|)\,dx$$

converges, and then by the convergence of $\int_a^{+\infty} |f(x)|\,dx$, we have

$$\int_a^{+\infty} f(x)\,dx = \int_a^{+\infty} (f(x) + |f(x)| - |f(x)|)\,dx = \int_a^{+\infty} (f(x) + |f(x)|)\,dx - \int_a^{+\infty} |f(x)|\,dx$$

converges too. ■

■ **Example 4.29** From Example 4.26, we know that

$$\int_3^{+\infty} \left| \frac{1}{(x-1)(x-2)} \right| dx$$

converges, so by Proposition 4.29, we conclude that

$$\int_3^{+\infty} \frac{\sin x}{(x-1)(x-2)}\,dx$$

converges too.

However, the converse of Proposition 4.29 is not true. It is possible that

$$\int_a^{+\infty} f(x)\,dx \text{ converges but } \int_a^{+\infty} |f(x)|\,dx \text{ diverges.}$$

In such case, we say $\int_a^{+\infty} f(x)\,dx$ *converges conditionally*.

Here is a counterexample:

■ **Example 4.30** Consider the improper integral:

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx.$$

For any $b > 1$, we have

$$\begin{aligned}\int_1^b \frac{\sin x}{x}\,dx &= -\int_1^b \frac{1}{x} d(\cos x) = -\left[\frac{\cos x}{x}\right]_1^b + \int_1^b \frac{\cos x}{-x^2}\,dx \\ &= \cos 1 - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx.\end{aligned}$$

As $b \to +\infty$, we have $\frac{\cos b}{b} \to 0$. For the integral of $\frac{\cos x}{x^2}$, we consider that

$$0 \le \left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}.$$

Since $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, by comparison test we know $\int_1^{+\infty} \left|\frac{\cos x}{x^2}\right| dx$ converges. Then by absolute convergence test,

$$\int_0^{+\infty} \frac{\cos x}{x^2}\,dx$$

converges as well. This shows

$$\lim_{b\to+\infty} \int_1^b \frac{\sin x}{x}\,dx$$

exists, and so $\int_1^{+\infty} \frac{\sin x}{x}\,dx$ converges.

However, one can show
$$\int_1^{+\infty} |\sin x|\, x\, dx$$
diverges. It suffices to find a sequence $\{b_n\}$ such that $b_n \to +\infty$ and
$$\lim_{n\to\infty} \int_1^{b_n} \left|\frac{\sin x}{x}\right| dx$$
diverges. We choose $b_n = n\pi$ (noting that $|\sin x|$ has period π). Then,
$$\begin{aligned}\int_1^{n\pi} \left|\frac{\sin x}{x}\right| dx &\geq \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \\ &\geq \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi}\, dx.\end{aligned}$$
By the periodicity of $|\sin x|$, one can easily see that
$$\int_{k\pi}^{(k+1)\pi} |\sin x|\ dx = \int_0^{\pi} |\sin x|\ dx,$$
so it is independent of k. Finally, we get
$$\int_1^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{\pi} \int_0^{\pi} |\sin x|\ dx \cdot \sum_{k=1}^{n-1} \frac{1}{k+1}.$$
As, $n \to +\infty$, $\sum_{k=1}^{n-1} \frac{1}{k+1}$ diverges to $+\infty$, hence
$$\lim_{n\to+\infty} \int_1^{n\pi} \left|\frac{\sin x}{x}\right| dx = +\infty.$$
This shows
$$\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx = \lim_{b\to+\infty} \int_1^{b} \left|\frac{\sin x}{x}\right| dx$$
diverges.

■ **Exercise 4.72** Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with period $T > 0$ (i.e $f(x+T) = f(x)$ for any $x \in \mathbb{R}$) and also that $f(x) \not\equiv 0$, show that $\displaystyle\int_1^{+\infty} \left|\frac{f(x)}{x}\right| dx$ diverges.

The method used to determine the convergence of the example $\displaystyle\int_a^{+\infty} \frac{\sin x}{x}\, dx$ can be further generalized. Consider the product $f(x)g(x)$ where f is continuous, and g is C^1 on $[a, +\infty)$. Let
$$F(x) := \int_a^x f(t)\, dt.$$
Then, by integration by parts, we have
$$\begin{aligned}\int_a^{+\infty} f(x)g(x)\, dx &= \int_a^{+\infty} g(x)\, d(F(x)) \\ &= \lim_{b\to+\infty} \left([F(x)g(x)]_a^b - \int_a^b F(x)g'(x)\, dx \right).\end{aligned}$$

If both $\lim_{b\to+\infty} F(b)g(b)$ and $\int_a^{+\infty} F(x)g'(x)\,dx$ converge, then $\int_a^{+\infty} f(x)g(x)\,dx$ converges too. These would happen if, for instance, one of the following conditions is met:

- $F(x)$ is bounded and $g'(x) \geq 0$ on $[a, +\infty)$, and $\lim_{x\to+\infty} g(x) = 0$ (known as Dirichlet Test);
- $\lim_{x\to+\infty} F(x)$ exists, $g'(x) \geq 0$ and g is bounded on $[a, +\infty)$ (known as Abel Test).

■ **Exercise 4.73** Verify that any of the above conditions implies that both $\lim_{b\to+\infty} F(b)g(b)$ and $\int_a^{+\infty} F(x)g'(x)\,dx$ converge.

■ **Example 4.31** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with period $T > 0$, such that

$$\int_0^T f(x)\,dx = 0.$$

Show that $\int_1^{+\infty} \frac{f(x)}{x}\,dx$ converges.

■ **Solution** For any $x > 1$, we consider let $k \in \mathbb{N}$ such that $1 + kT \leq x < 1 + (k+1)T$. Then,

$$F(x) := \int_1^x f(t)\,dt = \int_1^{1+T} f(t)\,dt + \int_{1+T}^{1+2T} f(t)\,dt + \cdots + \int_{1+(k-1)T}^{1+kT} f(t)\,dt + \int_{1+kT}^{x} f(t)\,dt.$$

By the periodicity of f, we know

$$\int_1^{1+T} f(t)\,dt = \int_{1+T}^{1+2T} f(t)\,dt = \cdots = \int_{1+(k-1)T}^{1+kT} f(t)\,dt = \int_0^T f(t)\,dt = 0,$$

so $F(x) = \int_{1+kT}^{x} f(t)\,dt$. Next we claim that $F(x)$ is bounded:

$$|F(x)| \leq \int_{1+kT}^{x} |f(t)|\,dt \leq \int_{1+kT}^{1+(k+1)T} |f(t)|\,dt = \int_0^T |f(t)|\,dt.$$

Here the last equality follows from the periodicity of the function $|f(t)|$. This shows $F(x)$ is bounded. Clearly $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2} \geq 0$, and $\left|-\frac{1}{x}\right| \leq 1$ on $[1, \infty)$. We can let $g(x) = -\frac{1}{x}$ then both F and g fulfill the conditions of the Abel test. Therefore, we conclude that

$$\int_1^{+\infty} \frac{f(x)}{-x}\,dx \qquad \text{and hence} \qquad \int_1^{+\infty} \frac{f(x)}{x}\,dx$$

converge.

■ **Exercise 4.74** Show that the following improper integrals converge:

$$\int_1^{+\infty} \frac{\sin x \tan^{-1} x}{x}\,dx \qquad \text{and} \qquad \int_1^{+\infty} \sin(x^2)\,dx.$$

■ **Exercise 4.75** Show that the following improper integral converges when $s > 0$:

$$\int_1^{\infty} \frac{x - [x]}{x^{s+1}}\,dx.$$

Show also that when $s > 1$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{s}{s-1} - s\int_1^{\infty} \frac{x-[x]}{x^{s+1}}\,dx.$$

[Remark 1: Although $x - [x]$ is discontinuous at integer points, it is a bounded function. There is no need to break down the above integral into $\int_1^2 + \int_2^3 + \int_3^4 + \cdots$ when showing the convergence, but it is necessary to do so when computing it.]

[Remark 2: Even though $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges when $s \in (0,1)$, the RHS

$$\frac{s}{s-1} - s\int_1^{\infty} \frac{x-[x]}{x^{s+1}}\,dx$$

is well-defined when $s \in (0,1)$. We can extend $\zeta(s)$ to a larger domain $(0,+\infty)\backslash\{1\}$ by giving it a more general definition as:

$$\hat{\zeta}(s) := \frac{s}{s-1} - s\int_1^{\infty} \frac{x-[x]}{x^{s+1}}\,dx$$

when $s \in (0,\infty)\backslash\{1\}$. Then, we would have $\hat{\zeta}(s) = \zeta(s)$ when $s > 1$. In complex analysis, one can show there is at most one way of extending a *holomorphic* function (beyond the scope of this course), so one usually simply writes ζ instead of $\hat{\zeta}$ for the extension. In fact, one can further extend ζ to the domain $\mathbb{C}\backslash\{1\}$ and show that the extended ζ takes the value $-\frac{1}{12}$ when $s = -1$. However, we have $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ only when $\mathrm{Re}(s) > 1$ and $\zeta(s)$ is defined as "something else" when $\mathrm{Re}(s) \leq 1$. It is totally a misconception that regard that $\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$.]

■ **Exercise 4.76** (a) Show that when $s > 1$, the improper integral $\displaystyle\int_0^{+\infty} \frac{t^{s-1}}{e^t - 1}\,dt$ converges.

(b) By a suitable substitution, show that for any $n \in \mathbb{N}$ and $s > 0$ we have

$$\frac{1}{n^s}\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-nt}\,dt.$$

Here Γ is the Gamma function defined in Exercise 4.70.

(c) Finally, show that

$$\zeta(s)\Gamma(s) = \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1}\,dt$$

for any $s > 1$, where $\zeta(s)$ is the zeta function.

[Remark 1: You may assume that it is legitimate to swapping $\displaystyle\sum_{n=1}^{\infty}$ with $\displaystyle\int_0^{+\infty}$. It is generally not always true, but we can justify it is fine in our example in MATH 3033/3043.]

[Remark 2: Hence we can take $\frac{1}{\Gamma(s)}\int_0^{+\infty} \frac{t^{s-1}}{e^t-1}\,dt$ as a new definition of $\zeta(s)$. With some further works (including the extension of $\Gamma(s)$), one can show that such an expression could give an extension to $\zeta(s)$ to the domain $\mathbb{C}\backslash\{1\}$.]