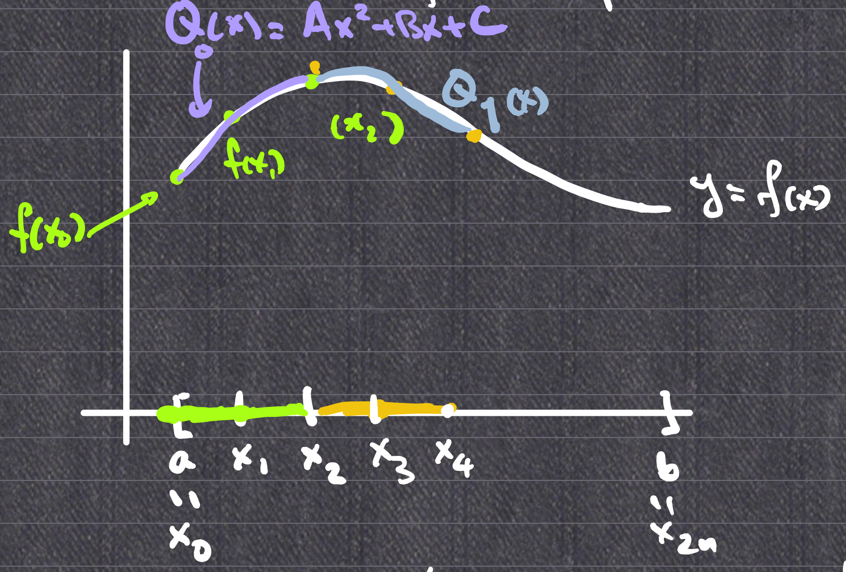
Simpson's rule.

- approximate f by quadratic functions.

$f: [a,b] \to \mathbb{R}$

$P_{2n}: \; a = x_0 < x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} = b.$

uniform partition.



Approximate $\displaystyle\int_a^b f(x)\,dx$ by $\displaystyle\sum_{i=0}^{n-1} \int_{x_{2i}}^{x_{2i+2}} Q_i(x)\,dx.$

<u>WANT:</u> $Q_i(x_{2i}) = f(x_{2i}), \quad Q_i(x_{2i+1}) = f(x_{2i+1}),$

$Q_i(x_{2i+2}) = f(x_{2i+2})$

$$Q_i(x) := f(x_{2i}) \cdot \frac{(x-x_{2i+1})(x-x_{2i+2})}{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})}$$

$$+ f(x_{2i+1}) \cdot \frac{(x-x_{2i})(x-x_{2i+2})}{(x_{2i+1}-x_{2i})(x_{2i+1}-x_{2i+2})}$$

$$+ f(x_{2i+2}) \cdot \frac{(x-x_{2i})(x-x_{2i+1})}{(x_{2i+2}-x_{2i})(x_{2i+2}-x_{2i+1})}$$

When $x = x_{2i+1}, x_{2i+2}$

$$\frac{(x-x_{2i+1})(x-x_{2i+2})}{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})} = 0.$$

When $x = x_{2i}$,

$$\frac{(x-x_{2i+1})(x-x_{2i+2})}{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})} = \frac{\cancel{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})}}{\cancel{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})}} = 1$$

FYI: such a Q_i is unique.

Let $P(x)$, $R(x)$ be quadratic functions such that $P(a) = R(a)$, $P(b) = R(b)$, $P(c) = R(c)$

then $P(x) \equiv R(x) \quad \forall x \in \mathbb{R}$. $\boxed{a<b<c}$

<u>Proof</u>: $P(x) = Ax^2 + Bx + C$

$R(x) = Ex^2 + Fx + G$.

$P(a) = R(a) \Rightarrow Aa^2 + Ba + C = Ea^2 + Fa + G$.

$\Rightarrow (A-E)a^2 + (B-F)a + (C-G) = 0$.

$$\begin{cases} (A-E)a^2 + (B-F)a + (C-G) = 0. \\ (A-E)b^2 + (B-F)b + (C-G) = 0 \\ (A-E)c^2 + (B-F)c + (C-G) = 0. \end{cases}$$

$$\Rightarrow \begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{bmatrix} \begin{bmatrix} A-E \\ B-F \\ C-G \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise:

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(b-c)(c-a) \neq 0.$$

$\therefore \quad A = E, \; B = F, \; C = G.$

Next: $\displaystyle\int_{x_{2i}}^{x_{2i}+2\Delta x} Q_i(x)\,dx = ?$

$$\Delta x = \frac{b-a}{2n} = x_{2i+1} - x_{2i} = x_{2i+2} - x_{2i+1}$$

$$\int_{x_{2i}}^{x_{2i}+2\Delta x} \frac{(x-x_{2i+1})(x-x_{2i+2})}{(x_{2i}-x_{2i+1})(x_{2i}-x_{2i+2})}\,dx$$

$$= \int_{x_{2i}}^{x_{2i}+2\Delta x} \frac{\overbrace{(x-x_{2i+1})}^{x-x_{2i}-\Delta x}(x-x_{2i+2})}{(-\Delta x)(-2\Delta x)}\,dx$$

let $y = x - x_{2i}$

$$= \int_0^{2\Delta x} \frac{(y-\Delta x)(y-2\Delta x)}{2(\Delta x)^2}\,dy$$

$$= \frac{1}{2(\Delta x)^2}\int_0^{2\Delta x} \left(y^2 - 3y\Delta x + 2(\Delta x)^2\right)dy$$

$$= \frac{1}{2(\Delta x)^2}\left[\frac{y^3}{3} - \frac{3y^2}{2}\Delta x + 2(\Delta x)^2 y\right]_0^{2\Delta x}$$

$$= \frac{1}{2(\Delta x)^2}\left(\frac{(2\Delta x)^3}{3} - \frac{3(2\Delta x)^2\Delta x}{2} + 2(\Delta x)^2\cdot 2\Delta x\right)$$

$$= \frac{\Delta x}{2}\left(\frac{8}{3} - \frac{3\cdot 2^2}{2} + 2\cdot 2\right) = \frac{\Delta x}{2}\left(\frac{2}{3}\right) = \frac{\Delta x}{3}.$$

$$\int_{x_{2i}}^{x_{2i+2}} Q_i(x)\,dx = \frac{\Delta x}{3}\left(f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right)$$

$$S_n = \sum_{i=0}^{n-1}\int_{x_{2i}}^{x_{2i+2}} Q_i(x)\,dx$$

Prop: Let $f:[a,b]\to\mathbb{R}$ be C^4.

$\exists\, C>0$ s.t.

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5}{Cn^4}\sup_{[a,b]}|f^{(4)}|.$$

Proof: On $[x_{2i}, x_{2i+2}]$:

$$f(x) = \underbrace{f(x_{2i}) + \frac{f'(x_{2i})}{1!}(x-x_{2i}) + \frac{f''(x_{2i})}{2!}(x-x_{2i})^2 + \frac{f'''(x_{2i})}{3!}(x-x_{2i})^3}_{P_i(x)}$$

$$+\frac{f^{(4)}(?)}{4!}(x-x_{2i})^4 \qquad \text{(Lagrange's remainder)}$$

Key idea: Bounding

$$I = \left|\int_{x_{2i}}^{x_{2i}+2\Delta x} f(x)\,dx - \int_{x_{2i}}^{x_{2i}+2\Delta x} P_i(x)\,dx\right|$$

$$\boxed{\begin{aligned}&|a-c|\\ &\le |a-b|\\ &\quad+|b-c|\end{aligned}}$$

and $$II = \left|\int_{x_{2i}}^{x_{2i}+2\Delta x} P_i(x)\,dx - \int_{x_{2i}}^{x_{2i}+2\Delta x} Q_i(x)\,dx\right|$$

$$I = \left|\int_{x_{2i}}^{x_{2i}+2\Delta x} \frac{f^{(4)}(?)}{4!}(x-x_{2i})^4\,dx\right| \le \int_{x_{2i}}^{x_{2i}+2\Delta x} \frac{|f^{(4)}(?)|}{4!}(x-x_{2i})^4\,dx$$

$$\le \int_{x_{2i}}^{x_{2i}+2\Delta x} \frac{\sup_{[a,b]}|f^{(4)}|}{4!}(x-x_{2i})^4\,dx$$

$$= \frac{\sup_{[a,b]}|f^{(4)}|}{4!}\left[\frac{(x-x_{2i})^5}{5}\right]_{x_{2i}}^{x_{2i}+2\Delta x}$$

$$= \frac{1}{4!}\sup_{[a,b]}|f^{(4)}|\cdot\frac{1}{5}(2\Delta x)^5 = \frac{2^5}{5!}(\Delta x)^5\sup_{[a,b]}|f^{(4)}|$$

$$\mathrm{II} = \left| \int_{x_{2i}}^{x_{2i}+2\Delta x} P_i(x)\,dx - \frac{\Delta x}{3}\Big(f(x_{2i}) + 4\underbrace{f(x_{2i+1})}_{f(x_{2i}+\Delta x)} + \underbrace{f(x_{2i+2})}_{f(x_{2i}+2\Delta x)} \Big) \right|$$

$$\int_{x_{2i}}^{x_{2i}+2\Delta x} P_i(x)\,dx$$

$$= \int_{x_{2i}}^{x_{2i}+2\Delta x} f(x_{2i}) + \frac{f'(x_{2i})}{1!}(x-x_{2i}) + \frac{f''(x_{2i})}{2!}(x-x_{2i})^2 + \frac{f'''(x_{2i})}{3!}(x-x_{2i})^3 \; dx$$

$$= \left[f(x_{2i})x + \frac{f'(x_{2i})}{1!}\cdot\frac{(x-x_{2i})^2}{2} + \frac{f''(x_{2i})}{2!}\cdot\frac{(x-x_{2i})^3}{3} + \frac{f'''(x_{2i})}{3!}\cdot\frac{(x-x_{2i})^4}{4} \right]_{x_{2i}}^{x_{2i}+2\Delta x}$$

$$= f(x_{2i})\cdot 2\Delta x + \frac{f'(x_{2i})}{2!}(2\Delta x)^2 + \frac{f''(x_{2i})}{3!}\cdot(2\Delta x)^3 + \frac{f'''(x_{2i})}{4!}(2\Delta x)^4.$$

$$= 2f(x_{2i})\Delta x + \frac{2^2}{2!}f'(x_{2i})(\Delta x)^2 + \frac{2^3}{3!}f''(x_{2i})(\Delta x)^3 + \frac{2^4}{4!}f'''(x_{2i})(\Delta x)^4$$

$$f(x_{2i}+\Delta x) = f(x_{2i}) + \frac{f'(x_{2i})}{1!}\Delta x + \frac{f''(x_{2i})}{2!}(\Delta x)^2 + \frac{f'''(x_{2i})}{3!}(\Delta x)^3$$

$$+ \frac{f^{(4)}(??)}{4!}(\Delta x)^4$$

$$f(x_{2i}+2\Delta x) = f(x_{2i}) + \frac{f'(x_{2i})}{1!}\cdot 2\Delta x + \frac{f''(x_{2i})}{2!}(2\Delta x)^2 + \frac{f'''(x_{2i})}{3!}(2\Delta x)^3$$

$$+ \frac{f^{(4)}(???)}{4!}(2\Delta x)^4$$

$$\int_{x_{2i}}^{x_{2i}+2\Delta x} Q_i(x)\,dx = \frac{\Delta x}{3}\Big(f(x_{2i}) + 4f(x_{2i}+\Delta x) + f(x_{2i}+2\Delta x) \Big)$$

$$= \frac{\Delta x}{3}\Big(6f(x_{2i}) + 6f'(x_{2i})\cdot\Delta x + \frac{1}{2!}(4+4)f''(x_{2i})(\Delta x)^2$$

$$+\frac{1}{3!}(4+2^3)\, f'''(\gamma_{2i})\cdot(\Delta x)^3$$

$$\frac{4\times 3}{3\times 2}=2$$

$$+\left(\frac{4 f^{(4)}(??)}{4!}+\frac{f^{(4)}(???)}{4!}\cdot 2^4\right)(\Delta x)^4 \Bigg)$$

$$\frac{2^4}{4!}=\frac{2\cdot 2\cdot 2\cdot 2}{4\cdot 3\cdot 2\cdot 1}=\frac{2}{3}.$$

$$\mathrm{II}=\left|\left(\frac{4 f^{(4)}(??)}{4!}+\frac{2^4}{4!} f^{(4)}(???)\right)(\Delta x)^4\cdot\frac{\Delta x}{3}\right|$$

$$\leq\left(\frac{4}{4!}\sup_{[a,b]}|f^{(4)}|+\frac{2^4}{4!}\sup_{[a,b]}|f^{(4)}|\right)\frac{(\Delta x)^5}{3}$$

$$=\frac{1}{3}\cdot\frac{4+2^4}{4!}\sup_{[a,b]}|f^{(4)}|\cdot(\Delta x)^5.$$

$$\Rightarrow\left|\int_{x_{2i}}^{x_{2i}+2\Delta x} f(x)-Q_i(x)\,dx\right|$$

$$\leq \mathrm{I}+\mathrm{II}$$

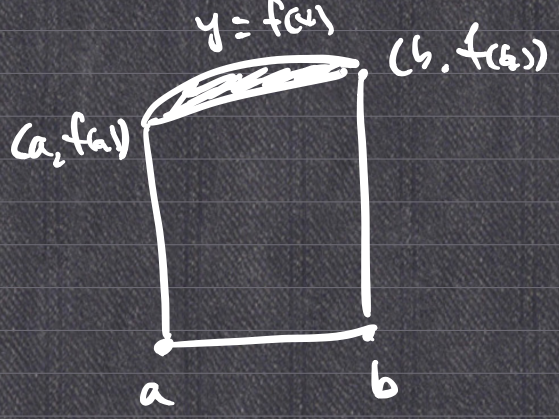
$$\Delta x=\frac{b-a}{2n}$$

$$=\left(\frac{2^5}{5!}+\frac{1}{3}\cdot\frac{4+2^4}{4!}\right)\sup_{[a,b]}|f^{(4)}|\cdot\frac{(b-a)^5}{2^5 n^5}$$

$$\therefore\left|\int_a^b f(x)\,dx-\sum_{i=0}^{n-1}\int_{x_{2i}}^{x_{2i}+2\Delta x} Q_i(x)\,dx\right|$$

$$\leq\frac{1}{2^5 n^4}\left(\frac{2^5}{5!}+\frac{1}{3}\cdot\frac{4+2^4}{4!}\right)\cdot(b-a)^5\sup_{[a,b]}|f^{(4)}|$$

$\blacksquare$



$$\int_a^b (f(x) - L(x))\,dx = \int_a^b f(x)\,dx - \int_a^b L(x)\,dx$$

$$= [x f(x)]_a^b - \int_a^b x f'(x)\,dx - \int_a^b L(x)\,dx$$

$$= [x f(x)]_a^b - \int_a^b f'(x)\,d\left(\frac{x^2}{2}\right) - \int_a^b L(x)\,dx$$

$$= [x \cdot f(x)]_a^b - \left[\frac{x^2}{2} f'(x)\right]_a^b + \int_a^b \frac{x^2}{2} f''(x)\,dx - \int_a^b L(x)\,dx$$