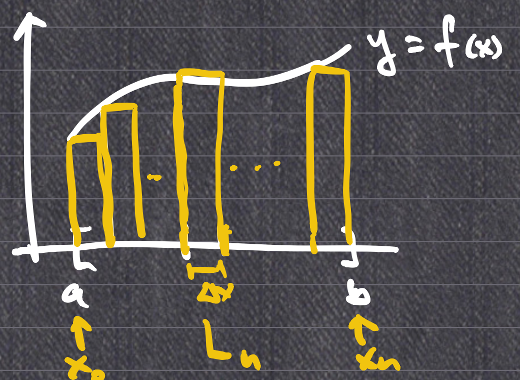
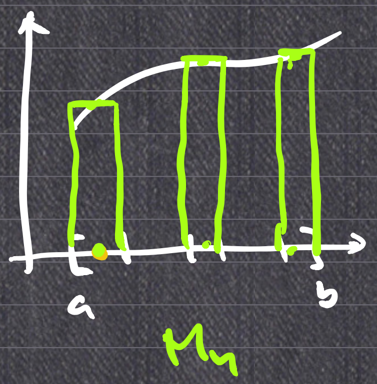




$$L_n = \underbrace{\Delta x}_{= \frac{b-a}{n}} \cdot \left(f(x_0) + \dots + f(x_{n-1})\right) = \sum_{i=0}^{n-1} f(a + i\Delta x) \cdot \Delta x.$$

$$M_n = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \cdot \Delta x$$

$$\int_1^2 e^{-x^2}\,dx \approx 0.1352\ldots$$

Last time:

$f : [a,b] \to \mathbb{R}$, C^1

then $$\left| \int_a^b f(x)\,dx - L_n \right| \leq \frac{(b-a)^2}{2n} \sup_{[a,b]} |f'|$$

Proof: $P_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$

$\Delta x = \frac{b-a}{n}$.

For each i,

$$\underline{\int_{x_{i-1}}^{x_i} f(x)\,dx} - \underbrace{f(x_{i-1})\,\Delta x}_{= \int_{x_{i-1}}^{x_i} f(x_{i-1})\,dx}$$

$$= \int_{x_{i-1}}^{x_i} \left(f(x) - f(x_{i-1})\right) dx$$

$$= \int_{x_{i-1}}^{x_i} f'(c(i,x)) \, (x - x_{i-1}) \, dx \quad , \exists\, c(i,x) \in [x_{i-1}, x]$$

$$\Rightarrow \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1})\Delta x \right| \le \int_{x_{i-1}}^{x_i} |f'(c(i,x))| \, (x - x_{i-1})\, dx$$

$$\le \int_{x_{i-1}}^{x_i} \sup_{[a,b]} |f'| \; (x - x_{i-1})\, dx$$

$$= \sup_{[a,b]} |f'| \cdot \left[\frac{x^2}{2} - x\, x_{i-1} \right]_{x_{i-1}}^{x_i}$$

$$= \sup_{[a,b]} |f'| \cdot \frac{(x_i - x_{i-1})^2}{2}$$

$$= \sup_{[a,b]} |f'| \cdot \frac{(b-a)^2}{2n^2} \qquad x_i - x_{i-1} = \frac{b-a}{n}$$

$$\left| \int_a^b f(x)\,dx - L_n \right| = \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_{i-1})\Delta x \right|$$

$$= \left| \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1})\Delta x \right) \right|$$

$$\le \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1})\Delta x \right| \le \sum_{i=1}^{n} \sup_{[a,b]} |f'| \frac{(b-a)^2}{2n^2}$$

$$= \sup_{[a,b]} |f'| \cdot \frac{(b-a)^2}{2n}$$

e.g.

$$\int_1^2 \underbrace{e^{-x^2}}_{f(x)} dx \qquad f'(x) = -2xe^{-x^2}$$

$$\sup_{[1,2]} |f'(x)| \le 2 \cdot 2 \cdot e^{-1} = \frac{4}{e}.$$

Take $n = 100$, then

$$\text{error} \le \sup_{[1,2]} |f'| \cdot \frac{(2-1)^2}{2n} = \frac{4}{e} \cdot \frac{1}{200} \approx 0.00735\ldots$$

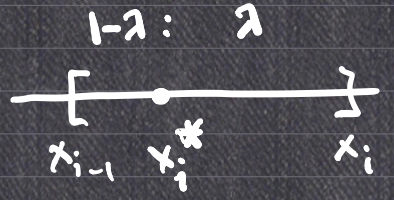
If we want error ≤ 0.00001,

$$\Uparrow$$

$$\underbrace{\sup_{[1,2]} |f'|}_{\frac{4}{e}} \cdot \frac{(2-1)^2}{2n} < 0.00001$$

$$\Uparrow$$

$$n > \frac{4}{e} \cdot \frac{1}{0.00001} \approx 147151.77\ldots$$



$\lambda \in [0,1]$

$$A_n := (\Delta x) \sum_{i=1}^{n} f(x_i^*).$$

min at $\lambda = \frac{1}{2}$.

Prop. 4.24:

$$\left| \int_a^b f(x)\,dx - A_n \right| \le \frac{(1-2\lambda+2\lambda^2)}{2n} \underbrace{(b-a)^2 \sup_{[a,b]} |f'|}$$

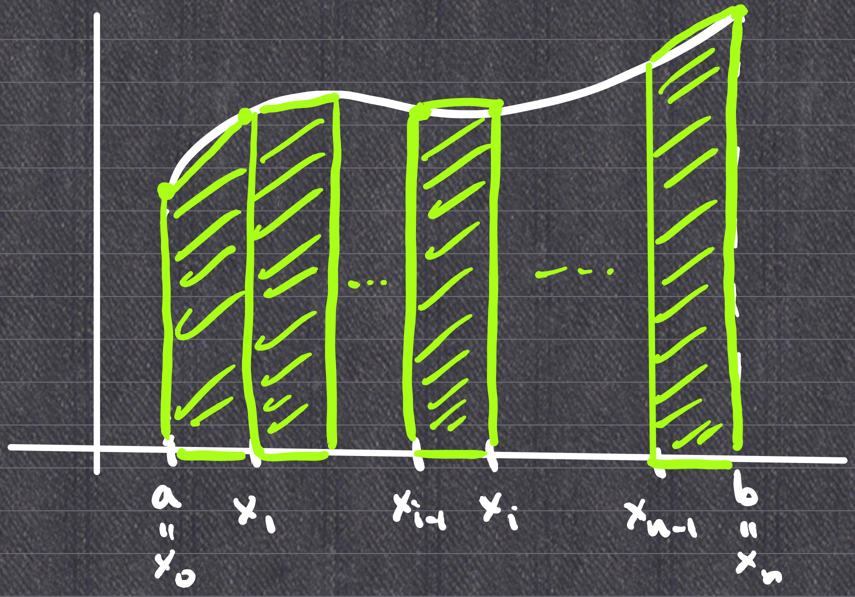
Proof outline:

$$\left| \int_{x_{i-1}}^{x_i} f(x) - \Delta x \cdot f(x_i^*) \right| = \left| \int_{x_{i-1}}^{x_i} \left(f(x) - f(x_i^*)\right) dx \right|$$

$$\le \int_{x_{i-1}}^{x_i} \left| f'(c_i(x)) \cdot (x - x_i^*) \right| dx$$

$$\le \sup_{[a,b]} |f'| \int_{x_{i-1}}^{x_i} |x - x_i^*| \, dx$$

Trapezoidal sum :



$$T_n = \sum_{i=1}^{n} \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) \Delta x$$

$$= \frac{1}{2} \Delta x \Big(f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n) \Big)$$

$$= \frac{1}{2} \Delta x \Big(f(a) + f(b) + 2\big(f(x_1) + \cdots + f(x_{n-1})\big) \Big)$$

$$= \frac{b-a}{n} \left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right) = \frac{L_n + R_n}{2} .$$

$$\left|\int_a^b f(x)\,dx - T_n\right| = \left|\frac{1}{2}\int_a^b f(x)\,dx - \frac{1}{2}L_n + \frac{1}{2}\int_a^b f(x)\,dx - \frac{1}{2}R_n\right|$$

$$\leq \frac{(b-a)^2}{2n}\sup_{[a,b]}|f'|.$$

Prop: $f:[a,b]\to\mathbb{R}$, C^2

then
$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{(b-a)^3}{12n^2}\sup_{[a,b]}|f''|$$

Proof: $x_i = a + i\Delta x$, $\Delta x = \frac{b-a}{n}$

$$A_i = -\frac{x_{i+1}+x_i}{2}, \quad B = -\frac{(\Delta x)^2}{4}.$$

$$\Big[(x+A_i)f(x)\Big]_{x_i}^{x_{i+1}} = \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \leftarrow$$

$$\Big[\big(\underbrace{(x+A_i)^2+B}_{(x-x_i)(x-x_{i+1})}\big)f'(x)\Big]_{x_i}^{x_{i+1}} = 0$$

$$\frac{d}{dx}\left((x+A_i)f(x) - \frac{1}{2}\big((x+A_i)^2+B\big)f'(x)\right)$$

$$= f(x) - \frac{1}{2}\big((x+A_i)^2+B\big)f''(x)$$

Newton-Leibniz

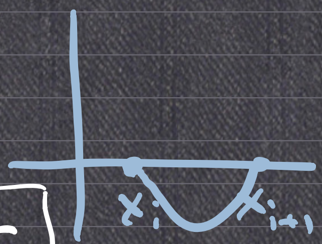
$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x) - \frac{1}{2}\big((x+A_i)^2+B\big)f''(x)\,dx$$

$$= \Big[(x+A_i)f(x) - \frac{1}{2}\big((x+A_i)^2+B\big)f'(x)\Big]_{x_i}^{x_{i+1}}$$

$$= \frac{f(x_i)+f(x_{i+1})}{2} \cdot \Delta x$$

$$\Rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\Delta x$$

$$= \frac{1}{2}\int_{x_i}^{x_{i+1}} \Big(\underbrace{(x+A_i)^2+B}_{(x-x_i)(x-x_{i+1})\,\leq\, 0.}\Big) f''(x)\,dx$$



$$\boxed{\int_{x_i}^{x_{i+1}} f(x)\,dx \begin{cases} \geq \dfrac{f(x_i)+f(x_{i+1})}{2}\Delta x & \text{if } f'' \leq 0 \\ \leq \quad - \quad - \quad - & \text{if } f'' \geq 0 \end{cases}}$$

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \right|$$

$$\leq \frac{1}{2}\int_{x_i}^{x_{i+1}} \left|(x-x_i)(x-x_{i+1})\, f''(x)\right| dx$$

$$\leq \frac{1}{2}\sup_{[a,b]}|f''| \cdot \int_{x_i}^{x_{i+1}} (x-x_i)(x_{i+1}-x)\,dx$$

$$= \frac{1}{2}\sup_{[a,b]}|f''| \cdot \frac{1}{6}(\Delta x)^3 \qquad . \quad \Delta x = x_{i+1} - x_i .$$

$$\left| \int_a^b f(x)\,dx - T_n \right| = \left| \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \right) \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \right|$$

$$\leq \sum_{i=0}^{n-1} \frac{1}{12}\sup_{[a,b]}|f''| \cdot \frac{(b-a)^3}{n^3} = \frac{(b-a)^3}{12n^2}\sup_{[a,b]}|f''|$$