Tutorial 5

Q1: Find all differentiable function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \int_0^{f(x)} e^{-t^2} dt$

By Fundamental thm of calculus

$$f'(x) = f'(x) e^{-(f(x))^2}$$

$\forall x \in \mathbb{R}$ either $f'(x) = 0$ or $f(x) = 0$

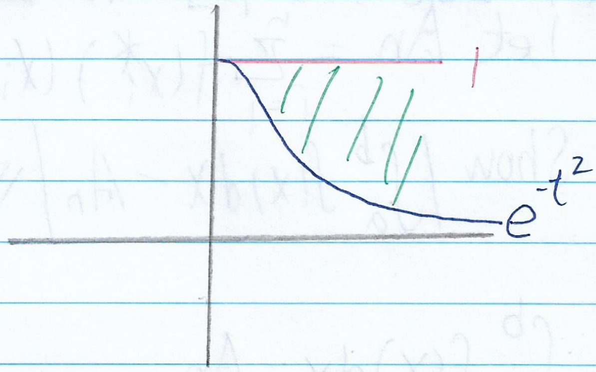
$$(f^2(x))' = 2f(x) \cdot f'(x) = 0$$

$$f^2(x) = C^2$$

$$\Rightarrow f = C \text{ or } f = -C$$

By continuity, f is const function

Solve $C = \int_0^C e^{-t^2} dt$



Let $F(y) = y - \int_0^y e^{-t^2} dt$

$$= \int_0^y (1 - e^{-t^2}) dt$$

$$> 0 \text{ if } y > 0$$

remark: try to integrate $\int_{-\infty}^{\infty} e^{-t^2} dt$

Q2 Let $f \in C^1[a,b]$

Let $L_n := \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$ where $x_i = a + \frac{(b-a)}{n} i$

$$\int_a^b f(x)dx - L_n = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) - f(x_{i-1}) \, dx$$

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (x - x_{i-1}) f'(a(x)) \, dx$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (x - x_{i-1}) dx \sup |f'(x)|$$

$$= \sum_{i=1}^{n} \frac{1}{2} (x - x_{i-1})^2 \sup |f'(x)|$$

$$= n \frac{1}{2} \frac{(b-a)^2}{n^2} \sup |f'(x)|$$

$$= \frac{1}{2} \frac{(b-a)^2}{n} \sup |f'(x)|$$

(exercise: show $L_n - \int_a^b f(x) dx \leq \frac{1}{2} \frac{(b-a)^2}{n} \sup |f'(x)|$

remark: The only difference of proof of L_n, R_n estimate is just

$$\underset{L_n}{\int_{x_{i-1}}^{x_i} (x - x_{i-1}) dx \geq 0}; \quad \underset{R_n}{\int_{x_{i-1}}^{x_i} (x - x_i) dx \leq 0}$$

Q3 By Q2 $\frac{(1-(-2))^2}{2n} \sup_{[-2,1]} |(\sin(x^2))'| \leq \frac{1}{100000}$

$|(\sin(x^2))'| = |2x\cos(x^2)| \leq 4$

$\Rightarrow n \geq \frac{100000}{9 \cdot 4} = 2777.777\ldots$

$\Rightarrow$ A choice is $n = 3000$

Q4 Let $f:[a,b] \to \mathbb{R}$ be a C^2 function on $[a,b]$

Let $A_n = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$ where $x_i^* = (1-\lambda)x_{i-1} + \lambda x_i$

Show $\left|\int_a^b f(x)dx - A_n\right| \leq \frac{|1-2\lambda|(b-a)^2}{2n} \sup_{[a,b]} |f'| + \frac{(1-3\lambda+3\lambda^2)(b-a)^3}{6n^2} \sup_{[a,b]} |f''|$

$\int_a^b f(x)dx - A_n$

$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) - f(x_i^*)$ write $f(x) = f(x_i^*) + f'(x_i^*)(x-x_i^*) + \frac{f''(\alpha(x))}{2}(x-x_i^*)^2$

$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f'(x_i^*)(x-x_i^*) + \frac{f''(\alpha(x))}{2}(x-x_i^*)^2$

independent of x

$= \sum_{i=1}^{n} f'(x_i^*)\left[\frac{1}{2}(x-x_i^*)^2\right]_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i} \frac{f''(\alpha(x))}{2}(x-x_i^*)^2$

$= \sum_{i=1}^{n} \frac{1}{2} f'(x_i^*)(1-2\lambda)(x-x_{i-1})^2 + \int_{x_{i-1}}^{x_i} \frac{f''(\alpha(x))}{2}(x-x_i^*)^2$

(just computation)

remark: we can do $\int_{x_{i-1}}^{x_i} f(x_i^*)(x-x_i^*) \leq \int_{x_{i-1}}^{x_i} |x-x_i^*| dx \sup |f|$

But we will give up more information

continuing the calculation

$$\int_a^b f(x)\,dx - A_n \le \frac{1}{2}\sup_{[a,b]}|f'|\,|1-2\lambda|\,\frac{(b-a)^2}{n} + \sum_{i=1}^{n}\frac{1}{2}\sup|f''|\underbrace{\int_{x_{i-1}}^{x_i}(x-x_i^*)^2}_{=\frac{1}{3}(1-3\lambda+3\lambda^2)(x_i-x_{i-1})^3}$$

$$= \frac{1}{2}\sup|f'|\,|1-2\lambda|\,\frac{(b-a)^2}{n}$$

$$+ \frac{1}{6}\sup|f''|(1-3\lambda+3\lambda^2)\frac{(b-a)^3}{n^2}$$

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remark: (for fun only)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx \qquad \begin{cases} y = r\cos\theta \\ x = r\sin\theta \end{cases}$$

$$= \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta \quad (r \text{ is from change of variable})$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,dr^2\,d\theta$$

$$\begin{aligned} &dy\,dx \\ &= d(r\cos\theta)\,d(r\sin\theta) \\ &= (r\sin\theta\,d\theta + \cos\theta\,dr)(r\cos\theta\,d\theta + \sin\theta\,dr) \\ &= r(\sin^2\theta+\cos^2\theta)\,d\theta\,dr \\ &= r\,d\theta\,dr \end{aligned}$$

$$= \int_0^{2\pi} -e^{-r^2}\Big|_0^{\infty}\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta = 2\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{2\pi}$$