Tutorial_5

Q1

Find all the differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \int_0^{f(x)} e^{-t^2} dt$

Q2

Proposition 4.23 Let f be a C^1 function on $[a, b]$. Then, the error between left-hand sum L_n and right-hand sum R_n (defined previously) and the actual integral $\displaystyle\int_a^b f(x)\,dx$ is bounded by:

$$\left| \int_a^b f(x)\,dx - L_n \right| \leq \frac{(b-a)^2}{2n} \sup_{[a,b]} |f'|$$

$$\left| \int_a^b f(x)\,dx - R_n \right| \leq \frac{(b-a)^2}{2n} \sup_{[a,b]} |f'|$$

■ **Exercise 4.57** Write up the proof of Proposition 4.23 using mean-value theorem instead.

■ **Exercise 4.58** Write up the proof of the right-hand sum part in Proposition 4.23. Clearly point out what are the essential differences from the proof of the left-hand sum.

Q3

■ **Exercise 4.59** Find an n such that the left-hand sum L_n gives an approximation of $\displaystyle\int_{-2}^{1} \sin(x^2)\,dx$ with accuracy up to 5 decimal places.

Q4

■ **Exercise 4.60 — Source: MATH1024 Spring 2018 Midterm.** Let $f : [a,b] \to \mathbb{R}$ be a C^2 function on $[a,b]$, and let A_n be as in Proposition 4.24. Show that:

$$\left| \int_a^b f(x)\,dx - A_n \right| \leq \frac{|1-2\lambda|\,(b-a)^2}{2n} \sup_{[a,b]} |f'| + \frac{(1-3\lambda+3\lambda^2)(b-a)^3}{6n^2} \sup_{[a,b]} |f''|\,.$$

[Hint: Consider second-order Taylor's approximation and its remainder.]

$$A_n := \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1}), \quad \text{where } x_i^* := (1-\lambda)x_{i-1} + \lambda x_i.$$