Basel Problem

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = ?$$

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$(s>1)$.

Proposed in 1644

First solved by Euler 1734

$$\sin x = 0 \iff x = n\pi \quad , n \in \mathbb{Z}.$$

$$\sin x = A\, x(\pi - x)(\pi + x)(2\pi - x)(2\pi + x) \cdots\cdots$$

$\uparrow$ to be determined

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x \to 0} A(\pi - x)(\pi + x)(2\pi - x)(2\pi + x) \cdots = 1$$

$$\Rightarrow A\, \pi^2 (2\pi)^2 (3\pi)^2 \cdots = 1$$

$$\Rightarrow A = \frac{1}{\pi^2 (2\pi)^2 (3\pi)^2 \cdots}$$

$$\sin x = \frac{1}{\underline{\pi^2}\, \underline{(2\pi)^2} (3\pi)^2 \cdots}\, x\, (\underline{\pi^2 - x^2})(\underline{(2\pi)^2 - x^2})((3\pi)^2 - x^2) \cdots$$

$$= x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right) \cdots$$

$$= x\left(1 - \left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots\right)x^2 + \cdots\cdots\right)$$

$$= x - \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)x^3 + \cdots.$$

$$\sin x = x - \frac{x^3}{6} + \cdots\cdots$$

Ex. 4.53

$$A_n = \int_0^{\pi/2} \cos^{2n} x \, dx \quad , \quad B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$$

Claim: $\quad 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) = \frac{1}{n^2} \qquad \forall n \in \mathbb{N}.$

Proof: $\quad A_n = \int_0^{\pi/2} \cos^{2n} x \, dx = \int_0^{\pi/2} \cos^{2n-1} x \cdot \cos x \, dx$

$$= \int_0^{\pi/2} \cos^{2n-1} x \, d(\sin x)$$

$$= \cancel{\left[\sin x \cos^{2n-1} x\right]_0^{\pi/2}} - \int_0^{\pi/2} \sin x \cdot (2n-1) \cos^{2n-2} x \cdot (-\sin x) \, dx$$

$$= \int_0^{\pi/2} (2n-1) \sin^2 x \cos^{2n-2} x \, dx$$

$$= \int_0^{\pi/2} (2n-1)(1-\cos^2 x) \cos^{2n-2} x \, dx$$

$$= (2n-1) A_{n-1} - (2n-1) A_n$$

$$\Rightarrow \quad A_n = \frac{2n-1}{2n} A_{n-1}$$

$$B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx = \int_0^{\pi/2} x^2 \cos^{2n-1} x \, d(\sin x)$$

$$= \left[x^2 \sin x \cos^{2n-1} x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, . \, d(x^2 \cos^{2n-1} x)$$

$$= - \int_0^{\pi/2} \sin x \left(2x \cos^{2n-1} x + x^2 . (2n-1) \cos^{2n-2} x . (-\sin x)\right)$$

$$= - \int_0^{\pi/2} 2x \underline{\sin x \cos^{2n-1} x \, dx} + (2n-1) \int_0^{\pi/2} x^2 \sin^2 x \cos^{2n-2} x \, dx$$

$$= \int_0^{\pi/2} \cancel{2}x \, d\left(\frac{1}{\cancel{2}n} \cos^{2n} x\right) + (2n-1) \int_0^{\pi/2} x^2 (1-\cos^2 x) \cos^{2n-2} x \, dx$$

$$= \cancel{\left[\frac{\cancel{2}x}{\cancel{2}n} \cos^{2n} x\right]_0^{\pi/2}} - \frac{1}{n} \int_0^{\pi/2} \cos^{2n} x \, dx + (2n-1)(B_{n-1} - B_n)$$

$$= -\frac{1}{n} A_n + (2n-1) B_{n-1} - (2n-1) B_n$$

$$\Rightarrow \quad 2n B_n = (2n-1) B_{n-1} - \frac{1}{n} A_n$$

$$\Rightarrow \quad 2n \frac{B_n}{A_n} = (2n-1) \frac{B_{n-1}}{A_n} - \frac{1}{n}$$

$$A_n = \frac{2n-1}{2n} A_{n-1}$$

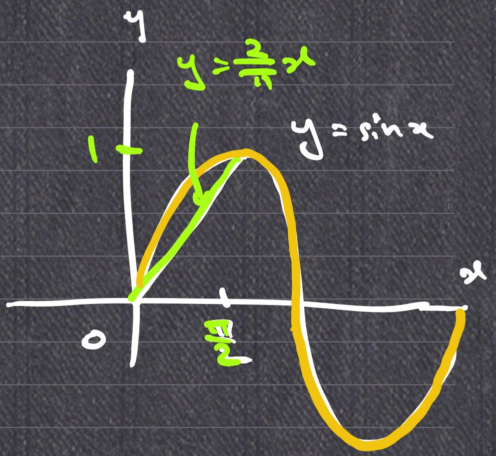
$$\Rightarrow \quad \frac{B_n}{A_n} = \frac{2n-1}{2n} \cdot \frac{B_{n-1}}{A_n} - \frac{1}{2n^2}$$

$$= \frac{B_{n-1}}{A_{n-1}} - \frac{1}{2n^2}.$$

(b) Claim: $\frac{B_n}{A_n} \le \frac{C}{n+1}$

Proof:

$\sin x \ge \frac{2}{\pi} x$ on $[0, \frac{\pi}{2}]$.



$$B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$$

$$\le \int_0^{\pi/2} \left(\frac{\pi}{2} \sin x\right)^2 \cos^{2n} x \, dx$$

$$= \left(\frac{\pi}{2}\right)^2 \int_0^{\pi/2} (1 - \cos^2 x) \cos^{2n} x \, dx$$

$$= \left(\frac{\pi}{2}\right)^2 (A_n - A_{n+1})$$

$$A_n = \frac{2n-1}{2n} A_{n-1}$$

$$= \left(\frac{\pi}{2}\right)^2 \left(A_n - \frac{2n+1}{2n+2} A_n\right)$$

$$= \left(\frac{\pi}{2}\right)^2 \left(\frac{1}{2n+2} A_n\right)$$

$$\Rightarrow \quad 0 \le \frac{B_n}{A_n} \le \frac{\pi^2}{4^2} \cdot \frac{1}{2(n+1)} \leftarrow C$$

(c)

$$2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) = \frac{1}{n^2} \qquad \text{from (a)}$$

From (b), $\displaystyle\lim_{n\to\infty} \frac{B_n}{A_n} = 0$.

$$\sum_{n=1}^{N} \frac{1}{n^2} = 2\sum_{n=1}^{N}\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right)$$

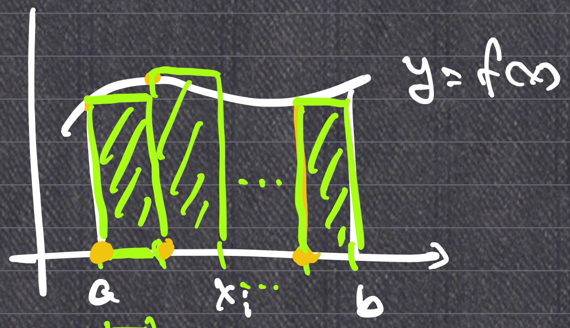
$$= 2\left(\frac{B_0}{A_0} \cancel{- \frac{B_1}{A_1}} \cancel{+ \frac{B_1}{A_1}} \cancel{- \frac{B_2}{A_2}} \cancel{+ \cdots} \cancel{+} \cancel{\frac{B_{N-1}}{A_{N-1}}} - \frac{B_N}{A_N}\right)$$

$$= 2\left(\frac{B_0}{A_0} - \frac{B_N}{A_N}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N\to\infty} 2\left(\frac{B_0}{A_0} - \frac{B_N}{A_N}\right) = \frac{2B_0}{A_0}$$

$$= 2\cdot\frac{\int_0^{\pi/2} x^2\,dx}{\int_0^{\pi/2} 1\,dx} = \frac{\pi^2}{6}.$$

§4.6 Numerical integrations.



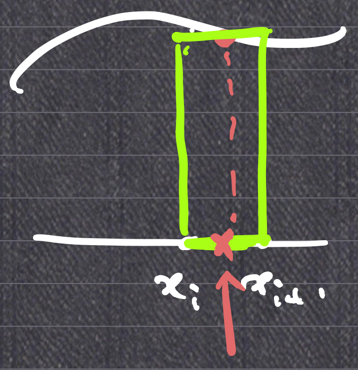
$\Delta x = \frac{b-a}{n}$

$$x_i = a + \frac{i(b-a)}{n}$$

$$L_n := \sum_{i=0}^{n-1} f(x_i)\cdot(x_{i+1} - x_i) = \sum_{i=1}^{n} f(x_{i-1})\underbrace{(x_i - x_{i-1})}_{\frac{b-a}{n}}$$

$$R_n := \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1}-x_i)$$

$$M_n := \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$



Prop: Given $f: [a,b] \to \mathbb{R}$ is C^1.

$$\Rightarrow \left| \int_a^b f(x)\,dx - \underset{\substack{\uparrow \\ \text{or } R_n}}{L_n} \right| \le \frac{(b-a)^2}{2n} \sup_{[a,b]} |f'|$$