Taylor's Remainder with integral form.

Given f is C^{n+1} on $I \ni a$:

$$f(x) = \underbrace{\sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j}_{T_n(x)} + R_n(x)$$

Lagrange remainder: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$\exists\, c$ between x and a.

Cauchy Remainder: $R_n(x) = \frac{f^{(n+1)}(c')}{n!}(x-c)^n(x-a)$

$\exists\, c'$ between x and a.

Integral Remainder:

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt$$

Proof:

$$I_n(x) := \frac{1}{n!}\int_a^x (x-t)^n \, d\left(f^{(n)}(t)\right)$$

$$= \frac{1}{n!}\left[(x-t)^n f^{(n)}(t)\right]_{t=a}^{t=x} - \frac{1}{n!}\int_a^x f^{(n)}(t)\cdot n(x-t)^{n-1}(-1)\, dt$$

$$= -\frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{\cancel{n!}_{(n-1)!}}\int_a^x \cancel{n}\, f^{(n)}(t)(x-t)^{n-1}\, dt$$

$$= -\frac{1}{n!}(x-a)^n f^{(n)}(a) + I_{n-1}(x)$$

$$I_n(x) - I_{n-1}(x) = -\frac{1}{n!} f^{(n)}(a) \cdot (x-a)^n. \quad \forall n \geq 1$$

$$I_n(x) = \left(I_n(x) - I_{n-1}(x)\right) + \left(I_{n-1}(x) - I_{n-2}(x)\right) + \cdots$$
$$+ \left(I_1(x) - I_0(x)\right) + I_0(x)$$

$$= -\frac{f^{(n)}(a)}{n!}(x-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} - \cdots$$
$$- \frac{f'(a)}{1!}(x-a)^1 + \underbrace{\frac{1}{0!}\int_a^x f'(t)\,dt}_{= f(x) - f(a)}$$

$$= -\sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j + f(x)$$

$$= R_n(x).$$

e.g. $e^x + e^{-x}$

$$= \sum_{j=0}^{2n} \frac{x^j}{j!} + \frac{1}{(2n)!}\int_0^x (x-t)^{2n} \underbrace{(e^t)^{(2n+1)}}_{e^t}\,dt$$

using the remainder formula for e^x.

$$+ \sum_{j=0}^{2n} \frac{(-1)^j x^j}{j!} + \frac{1}{(2n)!}\int_0^x (x-t)^{2n} \underbrace{(e^{-t})^{2n+1}}_{-e^{-t}}\,dt$$

... on e^{-t}.

$$= 2\left(1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}\right)$$
$$\underbrace{+ \frac{1}{(2n)!}\int_0^x (x-t)^{2n}\left(e^t - e^{-t}\right)dt}_{\text{next: estimate this}}.$$

$$\left|(e^{x}+e^{-x}) - 2\left(1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}\right)\right|$$

$$= \frac{1}{(2n)!}\left|\int_0^x (x-t)^{2n}(e^t-e^{-t})\,dt\right|$$

$$\le \frac{1}{(2n)!}\int_0^x \left|\underbrace{(x-t)^{2n}}_{\oplus}\underbrace{(e^t-e^{-t})}_{\oplus}\right|dt \qquad \left(\begin{array}{c}\text{Assume}\\ x>0\end{array}\right)$$

$t \in [0, x]$

$$= \frac{1}{(2n)!}\int_0^x \underbrace{(x-t)^{2n}}_{\le x^{2n}}(e^t-e^{-t})\,dt$$

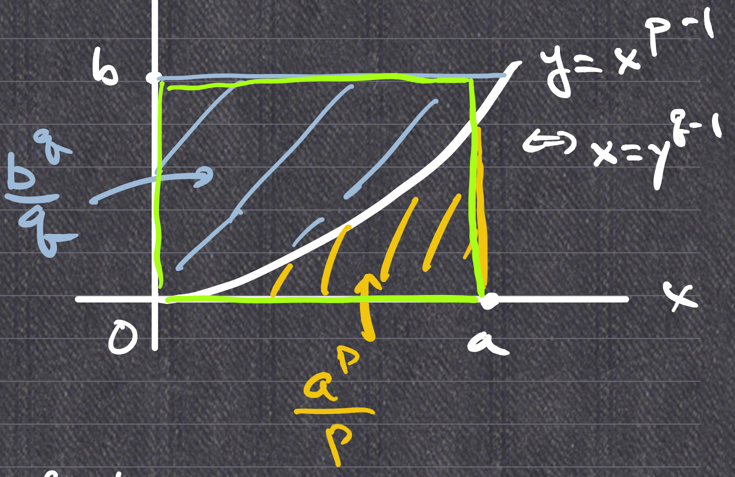
$$= \frac{x^{2n}}{(2n)!}\int_0^x (e^t-e^{-t})\,dt$$

$$= \frac{x^{2n}}{(2n)!}\Big[e^t+e^{-t}\Big]_{t=0}^{t=x} = \frac{x^{2n}}{(2n)!}(e^x+e^{-x}-2)$$

$ab \le \frac{a^p}{p}+\frac{b^q}{q}$ where $a, b>0$, $p, q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Young's inequality.

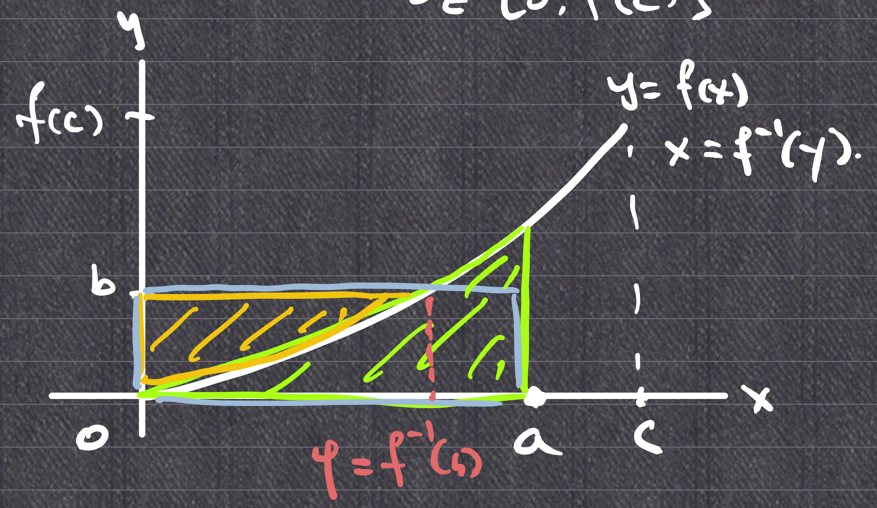
$\frac{1}{p} = 1-\frac{1}{q} \Rightarrow p = \frac{q}{q-1}$



$$y = x^{p-1} \Leftrightarrow x = y^{\frac{1}{p-1}}$$

$$= y^{\frac{1}{\frac{q}{q-1}-1}} = y^{q-1}.$$

Prop. 4.20 : Given $\begin{cases} f : [0, c] \to \mathbb{R} & \text{strictly increasing differentiable fct.} \\ a \in [0, c] \\ b \in [0, f(c)] & , \ f(0) = 0 \end{cases}$



then
$$\underbrace{\int_0^a f(x)\,dx}_{} + \underbrace{\int_0^b f^{-1}(y)\,dy}_{} \geq \underline{ab}.$$

<u>Proof</u> :

let $u = f^{-1}(y)$
$\Leftrightarrow f(u) = y$.

$$\int_0^b f^{-1}(y)\,dy$$

$$= \int_{u = f^{-1}(0) = 0}^{u = f^{-1}(b) = \varphi} u \, d(f(u))$$

$$= \Big[u f(u) \Big]_{u=0}^{u=\varphi} - \int_{u=0}^{u=\varphi} f(u)\,du$$

$$= \varphi f(\varphi) - \int_{x=0}^{x=\varphi} f(x)\,dx$$

$$= \varphi b - \int_0^\varphi f(x)\,dx$$

$$\Rightarrow \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$$

$$= \int_0^a f(x)\,dx + \varphi b - \int_0^\varphi f(x)\,dx$$

$$= \int_\varphi^a \underbrace{f(x)}_{\geq f(\varphi)\ (x\in[\varphi,a])}dx + \varphi b \qquad (\text{assume } \underline{\underline{\varphi \leq a}})$$

$$\geq \int_\varphi^a \underbrace{f(\varphi)}_{b}\,dx + \varphi b$$

$$= b(a-\varphi) + \varphi b = ab.$$

<u>HW3</u>: Case $\varphi > a$.

Hölder inequality

Given $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x)$ continuous on $[a,b]$.

then

$$\underbrace{\int_a^b |f(x)\,g(x)|\,dx}_{\|fg\|_1} \leq \underbrace{\left(\int_a^b |f(x)|^p\right)^{\frac{1}{p}}}_{=:\ \|f\|_p} \underbrace{\left(\int_a^b |g(x)|^q\right)^{\frac{1}{q}}}_{=:\ \|g\|_q}.$$

L^p-norm of f

Proof: $a = \dfrac{|f(x)|}{\|f\|_p}$, $b = \dfrac{|g(x)|}{\|g\|_q}$.

Young $\left(ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q} \right)$

$$\Rightarrow \frac{|f(x)|\,|g(x)|}{\|f\|_p \|g\|_q} \le \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^{\,p}} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^{\,q}}$$

$$\Rightarrow \int_a^b \frac{|f(x)|\,|g(x)|}{\|f\|_p \|g\|_q} dx \le \int_a^b \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^{\,p}} dx + \frac{1}{q} \int_a^b \frac{|g(x)|^q}{\|g\|_q^{\,q}} dx$$

$$= \frac{1}{p} \cdot \frac{1}{\cancel{\|f\|_p^{\,p}}} \cdot \cancel{\|f\|_p^{\,p}}$$

$$+ \frac{1}{q} \cdot \frac{1}{\cancel{\|g\|_q^{\,q}}} \cdot \cancel{\|g\|_q^{\,q}}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$