Tutorial 3

Q1: HKAL 1994 exercise 4.26

Let $f(x) = \int_1^x \sin(\cos t)\,dt$

(a) Show f is injective on $[0, \pi/2)$

(b) $\frac{d}{dx} f^{-1}(x)|_{x=0}$

pf: (a) $\cos t \in [0,1]$ $\forall t \in [0, \pi/2)$

$\sin(\cos t) \in [0,1]$ $\forall t \in [0, \pi/2)$

if $x > y$

$f(x) - f(y) = \int_y^x \sin(\cos t)\,dt > 0$

$\Rightarrow f(x) > f(y)$

f is strictly increasing function. Hence injective

(b) By Thm, let I be an interval and let $f: I \to \mathbb{R}$ be strictly monotone of I. Let $J = f(I)$ and $g: J \to \mathbb{R}$ be the function inverse to f.
If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$
then g is differentiable on J and $g' = \frac{1}{f' \circ g}$

remark: it is essential that $f'(c) \neq 0$

$g \circ f(x) = x$ $\quad g'(f(c)) f'(c) = 1$

but $f'(c) = 0$ $\Rightarrow 0 = 1$

pf of thm: Given $c \in \mathbb{R}$, $\exists \varphi$ s.t. φ is cont at c

$f(x) - f(c) = \varphi(x)(x - c)$ for $x \in I$ and $\varphi(c) = f'(c)$

namely: define $\varphi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & \text{for } x \neq c, x \in I \\ f'(c) \end{cases}$

It is easy to check $f(x) - f(c) = \varphi(x)(x - c)$
and φ is continuous at c

Since $f'(c) = \varphi(c) \neq 0$

$\exists V := (c - \delta, c + \delta)$ s.t. $\varphi(x) \neq 0$ $\forall x \in V \cap I$

let $U := f(V \cap I)$

$\forall y \in U$, $f(g(y)) = y$

Let $f(c) = d$

$y-d = f(g(y)) - f(c) = f(g(y)) - f(g(c))$

$= \varphi(g(y)) \cdot (g(y) - g(d))$

Note $\varphi(g(y)) \neq 0 \ \forall y \in U$

Hence $g(y) - g(d) = \frac{1}{\varphi(g(y))} \cdot (y-d)$

Since $1/\varphi \circ g$ is continuous at d,

$g'(d)$ exists and $g'(d) = 1/\varphi \circ g(d) = 1/\varphi(c) = 1/f'(c)$ //

(b) $f'(x) = \sin(\cos(x)) \neq 0$

$f'(g(0)) = \sin(\cos(1))$

$f(g(x)) = x$

$f'(g(x))g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$

$g'(0) = \frac{1}{f'(g(0))} = \frac{1}{\sin(\cos(1))}$

Q2 $\lim_{x \to 0^+} \left(\frac{1}{x^3} \int_0^x e^{t^2} dt - \frac{1}{x^2} \right)$

$= \lim_{x \to 0^+} \frac{\int_0^x e^{t^2} dt - x}{x^3}$

L'hospital rule $= \lim_{x \to 0^+} \frac{e^{x^2} - 1}{3x^2}$

L'hospital rule $= \lim_{x \to 0^+} \frac{2xe^{x^2}}{6x}$

$= \lim_{x \to 0^+} \frac{1}{3} e^{x^2} = \frac{1}{3}$

Q3: $\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x+a)(x-a)}dx$

(techinique of partial fraction) $= \frac{1}{2a}\int \left(\frac{-1}{x+a} + \frac{1}{x-a}\right)dx$

$= \frac{1}{2a}(-\log|x+a| + \log|x-a|) + C$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Partial fraction:

e.g. want to evaluate $\int \frac{p_1(x)}{p_2(x)}dx$ where p_1, p_2 are polynomial

rewrite $\frac{p_1(x)}{p_2(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{B_1}{(cx+d)} + \frac{C_1x+D_1}{cx^2+dx+e}$

To find A_1 A_2 A_3...

either by comparing coefficient or substition

e.g. sub factor, 0, i, 1 or other number

For $\int \frac{A_1}{ax+b}dx \Rightarrow \frac{A_1}{a}\ln|ax+b| + C$

$\int \frac{A_1}{(ax+b)^2} = -\frac{A_2}{a}\frac{1}{ax+b} + C$

$\int \frac{Bx+C}{cx^2+dx+e} = \int \frac{\alpha\frac{d}{dx}(cx^2+dx+e)}{cx^2+dx+e} + \int \frac{B\beta}{(x-\gamma)^2+\beta^2}dx$

$= \alpha\ln|cx^2+dx+e| + B\,\mathrm{arc}\left(\frac{x-\gamma}{\beta}\right)$

remark $\int \frac{1}{x^2+1}dx = \arctan x + C$

Q3: $\int \frac{1}{\sqrt{x^2+a^2}}$ Q4 $\int_0^r \sqrt{r^2-x^2}\,dx$

general strategy: if want to integrate $\frac{1}{\sqrt{1-x^2}}, \frac{1}{\sqrt{1+x^2}}, \sqrt{1+x^2}, \sqrt{1-x^2}$

always try to sub trigo or hypo trigo

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cosh^2 x - \sinh^2 x = 1$$

where $\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

e.g. $\int \sqrt{1-x^2}\,dx$ for simplicity assume the domain is s.t. $\cos\theta \geq 0$

Let $x = \sin\theta$

$$= \int \sqrt{1-\sin^2\theta}\,d\sin\theta = \int \cos^2\theta\,d\theta = \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$
$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}\sin\theta\cos\theta + C$$
$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

e.g. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{1}{\sin\theta}\,d\cos\theta \quad (\text{let } x = \cos\theta)$$
$$= -\int d\theta = -\theta + C = -\cos^{-1}\theta + C$$

[redacted]

$$\begin{cases} \dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}} \\ \dfrac{d}{dx} \arccos x = -\dfrac{1}{\sqrt{1-x^2}} \\ \dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2} \end{cases} \qquad \begin{cases} \dfrac{d}{dx} \operatorname{arc\,csc} x = -\dfrac{1}{|x|\sqrt{x^2-1}} \\ \dfrac{d}{dx} \operatorname{arc\,sec} x = \dfrac{1}{|x|\sqrt{x^2-1}} \\ \dfrac{d}{dx} \cot x = -\dfrac{1}{1+x^2} \end{cases}$$

Q5 $\int_a^b x\cos(x^2+1)\,dx$

$= \frac{1}{2}\int_a^b \cos(x^2+1)\,d(x^2+1)$

$= \frac{1}{2}\sin(x^2+1)\Big|_{x=a}^{x=b} = \frac{1}{2}\sin(b^2+1) - \frac{1}{2}\sin(a^2+1)$

$\int_a^b x^3 e^{x^4}\,dx = \int_a^b e^{x^4}\,dx^4 = e^{x^4}\Big|_{x=a}^{x=b} = e^{b^4} - e^{a^4}$

$\int_a^b \frac{x}{1+x^2}\,dx = \int_a^b \frac{1}{1+x^2}\,d(1+x^2) = \ln(1+x^2)\Big|_a^b$

$= \ln(1+b^2) - \ln(1+a^2)$

Q6 $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{\sin x}\,d\sin x = \log|\sin x| + C$

$\int \csc x\,dx = \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx = -\int \frac{1}{\csc x + \cot x}\,d(\csc x + \cot x)$

$= -\log|\csc x + \cot x| + C$

Other:

e.g. $\int \frac{x^n + x^{n-1} + \cdots}{\sqrt{x \pm a}} \Rightarrow$ sub $\sqrt{x \pm a}$

e.g. $\int \frac{x^n}{\sqrt{1 \pm x^2}}$ or $\int x^n \sqrt{1+x^2}$ if n is odd and positive $\Rightarrow d\sqrt{1+x^2}$

e.g. $\int (\cosh x)^n dx$ or $\int (\sinh x)^n dx$

if n is odd & positive $\Rightarrow d\sinh(x)$ or $d\cosh(x)$

then $1 + \sinh^2\theta = \cosh^2\theta$