§4.5 – Integration by parts

$f, g : [a,b] \to \mathbb{R}$, C^1 functions

$$\int_a^b f(x)\, g'(x)\, dx = [f(x)\, g(x)]_a^b - \int_a^b g(x)\, f'(x)\, dx$$

<u>Proof:</u> $\frac{d}{dx}(f(x)\, g(x)) = f'(x)\, g(x) + g'(x)\, f(x)$.

Newton-Leibniz $\Rightarrow$ $\int_a^b f'(x)\, g(x) + f(x)\, g'(x)\, dx$

$= [f(x)\, g(x)]_a^b$

$\Rightarrow$ $\int_a^b f(x)\, g'(x)\, dx = [f(x)\, g(x)]_a^b - \int_a^b f'(x)\, g(x)\, dx$ ∎

Symbolically:

$g'(x)\, dx = d(g(x))$

$f'(x)\, dx = d(f(x))$

$\int_a^b f(x)\, d(g(x))$

$= [f(x)\, g(x)]_a^b - \int_a^b g(x)\, d(f(x))$

<u>e.g.</u> $\int_1^2 \underbrace{\log x}_{f(x)}\, \underbrace{dx}_{g(x)} = [x \log x]_1^2 - \int_1^2 x (\log x)'\, dx$

$= 2\log 2 - \int_1^2 \underbrace{\cancel{x} \cdot \frac{1}{\cancel{x}}}_{1}\, dx$

$= 2\log 2 - 1$

$\int_1^2 \log x = \int_1^2 \underbrace{\log x}_{f} \cdot \underbrace{\frac{dx^{\,g}}{dx}}_{g'}$

e.g. $\int \sec^3 x\, dx = \int \sec x \cdot \underbrace{\sec^2 x\, dx}_{d(\tan x)} = \int \sec x\, d(\tan x)$

du when $u = \tan x$.

$= \sec x \tan x - \int \tan x\, d(\sec x)$

$= \sec x \tan x - \int \tan^2 x \sec x\, dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx.$

$\Rightarrow 2 \int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$ ✓

$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x\, dx$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \log|\sec x + \tan x| + C.$

Irrationality of e

$I_n := \int_0^1 x^n e^{-x}\, dx \quad (n \geq 0)$

Claim: $I_n = \frac{n!}{e}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$

Proof: $I_{n+1} = \int_0^1 x^{n+1} e^{-x}\, dx = \int_0^1 x^{n+1} \underbrace{d(-e^{-x})}_{e^{-x}dx}$

$= \left[-e^{-x} x^{n+1}\right]_0^1 + \int_0^1 e^{-x}\, d(x^{n+1})$

$= \quad -\frac{1}{e} \quad + (n+1) \int_0^1 x^n e^{-x}\, dx$

$$I_{n+1} = -\frac{1}{e} + (n+1) I_n$$

$$\Rightarrow \quad \frac{I_{n+1}}{(n+1)!} = -\frac{1}{e(n+1)!} + \frac{1}{n!} I_n$$

$$J_n := \frac{I_n}{n!} \quad \Rightarrow \quad J_{n+1} - J_n = -\frac{1}{e\cdot(n+1)!}$$

$$J_n = (J_n - J_{n-1}) + (J_{n-1} - J_{n-2}) + \cdots + (J_1 - J_0) + J_0$$

$$= -\frac{1}{e}\left(\frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + \frac{1}{1!}\right) + \int_0^1 e^{-x}\,dx$$

$$\Rightarrow \quad \frac{I_n}{n!} = -\frac{1}{e}\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + \underbrace{\left[-e^{-x}\right]_0^1}_{-\frac{1}{e}+1}$$

$$= -\frac{1}{e}\Big(\underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}}_{\sum_{k=0}^{n} \frac{1}{k!}} - e\Big)$$

$$\Rightarrow \quad I_n = \frac{n!}{e}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right). \qquad \blacksquare$$

$$\int_0^1 x^n e^{-x}\,dx = \frac{n!}{e}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

$$0 \le \int_0^1 \underbrace{x^n e^{-x}}_{f(x)}\,dx \underset{\wedge}{<} \int_0^1 \frac{1}{e}\,dx = \frac{1}{e}.$$

$$f'(x) = n x^{n-1} e^{-x} + x^n(-e^{-x}) \quad \Rightarrow \quad f(x) \le f(1) \text{ on } [0,1]$$

$$= e^{-x} x^{n-1} (n - x) > 0 \qquad = \frac{1}{e}.$$

(when $n \in \mathbb{N}$) on $x \in [0,1]$.

$$\frac{1}{e} > I_n = \frac{n!}{e}\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right) > 0 \quad I_n > 0 \text{ by } \underset{\text{on } (0,1)}{\underset{>0}{x^n e^{-x}}}$$

$$\Rightarrow \quad 1 > n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right) > 0$$

Suppose $e = \frac{m}{n}$, $m, n \in \mathbb{N}$.

$$\Rightarrow n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right) = n!\left(\frac{m}{n} - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right)$$

$$\in \mathbb{Z}.$$

But $n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right) \in (0,1)$

$\cancel{\rightarrow\leftarrow}$.

e.g. $I_{m,n} = \int \cos^m x \sin^n x \, dx$

Claim: $I_{m,n} = -\frac{1}{m+n} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+n} I_{m,n-2}$ $\quad \forall m \geq 0, n \geq 2.$

Proof:

$$I_{m,n} = \int \cos^m x \sin^n x \, dx$$

$$= \int \cos^m x \cdot \sin^{n-1} x \cdot \sin x \, dx$$

$$= \int \cos^m x \cdot \sin^{n-1} x \, d(-\cos x)$$

$$= -\cos^{m+1} x \sin^{n-1} x + \int \cos x \, d(\cos^m x \cdot \sin^{n-1} x)$$

$$= -\cos^{m+1} x \cdot \sin^{n-1} x + \int \cos x \left(m \cos^{m-1} x (-\sin x) \cdot \sin^{n-1} x + (n-1) \sin^{n-2} x \cos x \cdot \cos^m x\right)$$

$$= -\cos^{m+1} x \cdot \sin^{n-1} x + \int \left(\underbrace{-m \cos^m x \sin^n x}_{-m I_{m,n}} + (n-1) \cos^{m+2} x \sin^{n-2} x\right)$$

$$(m+1) I_{m,n} = -\cos^{m+1}x \cdot \sin^{n-1}x + (n-1)\int \cos^{m+2}x \, \sin^{n-2}x$$

$$= -\cos^{m+1}x \cdot \sin^{n-1}x + (n-1)\int \cos^{m}x \cdot \underbrace{(1-\sin^2 x)}_{\cos^2 x} \cdot \sin^{n-2}x \, dx$$

$$= -\cos^{m+1}x \cdot \sin^{n-1}x + (n-1) I_{m,n-2} - \underbrace{(n-1) I_{m,n}}$$

$$\int \cos^4 x \, \sin^6 x \, dx = I_{4,6} = -\frac{1}{10}\cos^5 x \, \sin^5 x + \frac{5}{10} I_{4,4}$$

$$I_{4,4} = -\frac{1}{8}\cos^5 x \, \sin^3 x + \frac{3}{8} I_{4,2}$$

$$I_{m,n} = -\frac{1}{m+n}\cos^{m+1}x \, \sin^{n-1}x + \frac{n-1}{m+n} I_{m,n-2}$$

$$\forall m \geq 0, \; n \geq 2.$$

$$I_{n,1} = \int \cos^n x \, \underbrace{\sin x \, dx}_{d(-\cos x)}$$

e.g. $\quad I_n := \int_0^1 x^n \sqrt{1-x}\, dx$

$$I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}} dx = \int_0^1 x^n \, d\left(-\frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}}\right)$$

$$= \left[-\frac{2}{3} x^n (1-x)^{\frac{3}{2}}\right]_0^1 + \int_0^1 \frac{2}{3}(1-x)^{\frac{3}{2}} \, d(x^n)$$

$$= 0 + \frac{2n}{3} \int_0^1 x^{n-1} (1-x)^{\frac{3}{2}} dx$$

$$= \frac{2n}{3} \int_0^1 x^{n-1} (1-x)^{\frac{1}{2}} (1-x) \, dx$$

$$= \frac{2n}{3} \int_0^1 x^{n-1} (1-x)^{\frac{1}{2}} - x^n (1-x)^{\frac{1}{2}} \, dx$$

$$I_n = \frac{2n}{3}\left(I_{n-1} - I_n\right) \Rightarrow \boxed{I_n = \frac{2n}{2n+3} I_{n-1}}$$