If $F' = f$ where F is C^1 on $[a,b]$.

then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a).$

Indefinite integral

$$\int f(x)\,dx = \{ F : F' = f \} = \{ F_0 + C : C \text{ is any real constant} \}.$$

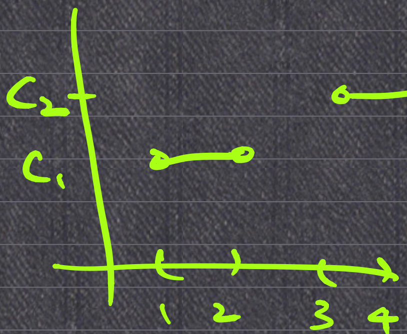
If $F_0' = f$

$$\int f(x)\,dx = F_0(x) + C.$$

$$f(x) = \begin{cases} \sin x & \text{if } x \in (1,2) \\ \cos x & \text{if } x \in (3,4) \end{cases}$$

$f: \underset{1 \quad 2}{(\quad)}\ \underset{3 \quad 4}{(\quad)} \to \mathbb{R}$

set of anti-derivatives on $(1,2) \cup (3,4)$ $= \begin{cases} -\cos x + C_1 & \text{if } x \in (1,2) \\ \sin x + C_2 & \text{if } x \in (3,4) \end{cases}$



where C_1, C_2 are any real constant.

Whenever we talk about indefinite integrals we will implicitly assume domain of f is an interval

$$\int \frac{1}{x}\,dx = \log x + C \text{ on interval } (a,b) \text{ where } a \geq 0.$$

$$\frac{d}{dx} \log x = \frac{1}{x}. \quad x > 0.$$

$$\int \frac{1}{x}\,dx = \log(-x) + C \text{ on interval } (a,b) \text{ where } b \leq 0.$$

$\log(-x)$ $\qquad$ $\log x$

0

$$\int \frac{1}{x}\,dx = \log|x| + C$$

Implicit assumption: interval $\not\ni 0$

$$\int_1^2 \frac{1}{x}\,dx = \Big[\log|x|\Big]_1^2 = \log 2$$

$$\int_{-2}^{-1} \frac{1}{x}\,dx = \Big[\log|x|\Big]_{-2}^{-1} = -\log 2$$

But: $\underline{\int_{-1}^{2} \frac{1}{x}\,dx} \neq \Big[\log|x|\Big]_{-1}^{2} = \log 2$

§4.4

e.g. $\int_0^2 x(2x^2+3)^2\,dx$

let $u = 2x^2+3$, $\quad du = d(2x^2+3) = \underline{4x\,dx}$

(by the virtue that $\frac{du}{dx} = 4x$)

When $x=0$, $u=3$

when $x=2$, $u=11$.

$$\int_0^2 \underline{x}(2x^2+3)^2\,\underline{dx} = \frac{1}{4}\int_0^2 \underbrace{(2x^2+3)}_{u}{}^2 \cdot \underbrace{4x\,dx}_{du}$$

$$= \frac{1}{4}\int_3^{11} u^2\,du = \checkmark$$

Prop 4.13: $\quad f$ is a continuous on $[g(a), g(b)]$.

g is C^1 on $[a,b]$

$$\Rightarrow \int_a^b f(\underset{\sim}{g(x)})\, \underbrace{g'(x)\, dx}_{du} = \int_{g(a)}^{g(b)} f(u)\, du \longleftarrow (*).$$

let $u = g(x)$

$$\boxed{\begin{aligned} u &= g(x) \\ du &= g'(x)\, dx \end{aligned}}$$

<u>Proof</u>: $F(x) = \int_{g(a)}^{x} f(u)\, du \rightsquigarrow F'(x) = f(x).$

$$\frac{d}{dx} F(g(x)) = \left.\frac{dF}{dx}\right|_{g(x)} \left.\frac{dg}{dx}\right|_{x} = F'(g(x))\, g'(x)$$
$$= f(g(x))\, g'(x)$$

$$\Rightarrow \int_a^b f(g(x))\, g'(x)\, dx = \Big[F(g(x))\Big]_{x=a}^{x=b}$$
$$= \underline{F(g(b)) - F(g(a))}$$

$$\int_{g(a)}^{g(b)} f(u)\, du = \Big[F(u)\Big]_{u=g(a)}^{u=g(b)} = F(g(b)) \underline{- F(g(a))}$$

$F'(u) = f(u).$

$\boxtimes$

$$\int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du$$

$$g'(x)\, dx = du$$

let $u = g(x)$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

dx dy dz dt ?

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = \int \frac{-du}{u} = -\log|u| + C$$

$$= -\log|\cos x| + C$$

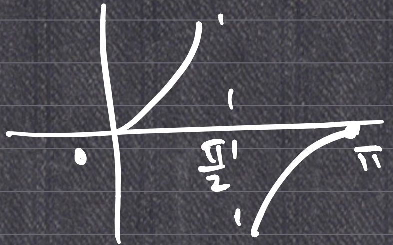
let $u = \cos x$,

$du = -\sin x\,dx$

$$= \log\frac{1}{|\cos x|} + C$$

$$= \log|\sec x| + C$$

$$\cancel{\int_0^{\pi} \tan x\,dx = \Big[\log|\sec x|\Big]_0^{\pi} = \log 1 - \log 1 = 0}$$



$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

$$= \int \frac{\overbrace{\sec^2 x}^{\frac{d}{dx}\tan x} + \overbrace{\sec x \tan x}^{=\frac{d}{dx}\sec x}\,dx}{\underbrace{\sec x + \tan x}_{u}} \; = du$$

$$= \int \frac{1}{u}\,du = \log|u| + C$$

let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)\,dx$

$$= \log|\sec x + \tan x| + C$$

$a > 0$

defined on $(-a, a)$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2 u}}\,a\cos u\,du = \int \frac{\cancel{a\cos u}}{\sqrt{a^2\cos^2 u}}\,du = \int du$$

$$= u + C$$

$= |a\cos u|$

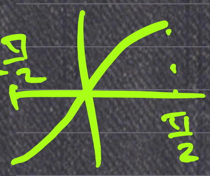
$= \cancel{a\cos u}$

$\sqrt{x^2} = |x|$

$$= \sin^{-1}\frac{x}{a} + C.$$

let $x = a\sin u$, $dx = a\cos u\,du$.

(implicitly means: let $u = \sin^{-1}\frac{x}{a}$)



$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \cos u \geq 0$

$$\int \frac{1}{a^2+x^2}\,dx$$

$\uparrow$

let $x = a\tan u \Rightarrow dx = a\sec^2 u\,du$

$\left(u = \tan^{-1}\frac{x}{a}\right)$

$$= a(1+\tan^2 u)\,du$$

$$= a\left(1+\frac{x^2}{a^2}\right)du$$

$$= \frac{1}{a}(a^2+x^2)\,du$$

$$\frac{1}{a^2+x^2}dx = \frac{1}{a}du \quad \leftarrow$$

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a}du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx =$$

$\uparrow$

$x = a\tan u$

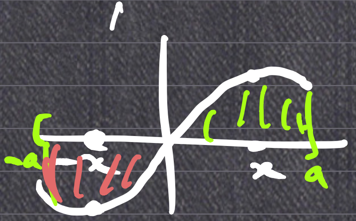
$$\int \frac{1}{x^2-a^2}\,dx =$$

$\uparrow$

$x = a \cdot\cdot\cdot$

Let f be an odd function: $f(-x) = -f(x)$ $\forall x \in \mathbb{R}$.

then: $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$



Proof: It suffices to show:

$$\underbrace{\int_{-a}^{0} f(x)\,dx} = -\underbrace{\int_{0}^{a} f(x)\,dx}$$

let $u = -x$, $du = -dx$

when $x = -a$, $u = a$

when $x = 0$, $u = 0$.

$$\int_{x=-a}^{x=0} f(x)\,dx = \int_{u=a}^{u=0} f(\underbrace{-u}_{x})\,\underbrace{(-du)}_{dx} = \int_a^0 \underbrace{-f(u)}_{f(-u)}\,(-du)$$

$$= \int_a^0 f(u)\,du = -\int_{u=0}^{u=a} f(u)\,du.$$

$$= -\int_{x=0}^{x=a} f(x)\,dx$$

If f is an even function, $f(-x)=f(x)\ \forall x\in\mathbb{R}$.

then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

$f(x+T)=f(x)\ \forall x\in\mathbb{R}$

$$f(x+T)=f(x) \quad \forall x\in\mathbb{R}.$$

(a) Prove: $\int_{a+kT}^{b+kT} f(x)e^{-x}\,dx = e^{-kT}\int_a^b f(x)e^{-x}\,dx.$

Solution: let $u=x+kT$, $du=dx$

when $x\in[a,b] \Rightarrow u\in[a+kT, b+kT]$.

$$\int_a^b f(x)e^{-x}\,dx = \int_{a+kT}^{b+kT} \underbrace{f(u-kT)}_{f(u)}\,e^{-(u-kT)}\,du$$

$\uparrow$

$x=u-kT$

constant.

$$= \int_{a+kT}^{b+kT} e^{-u}\,e^{kT} f(u)\,du$$

$f(u-kT)$

$= f(u-kT+\underbrace{T+T+\dots+T}_{k})$

$= f(u)$

$$e^{-kT}\int_a^b e^{-x}f(x)\,dx = \int_{a+kT}^{b+kT} f(u)e^{-u}\,du$$

$$= \int_{a+kT}^{b+kT} f(x)\, e^{-x}\, dx$$