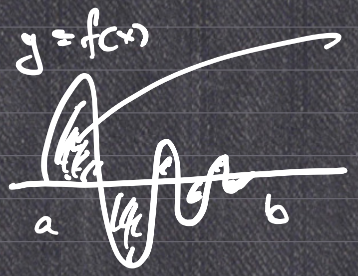


Jordan measurable $\overset{\text{def}}{\Longleftrightarrow}$ f is Riemann integrable on $[a,b]$.

$$\int_a^b f(x)\,dx.$$

$\int_0^1 x^p\,dx \quad (p \in \mathbb{N}).$ $\qquad 1^p + 2^p + 3^p + \cdots + n^p = ?$

$\int_0^1 e^x\,dx, \quad \int_0^\pi \sin x\,dx, \ \ldots$

Newton-Leibniz formula:

Given f is continuous on $[a,b]$ and $\exists\, F : [a,b] \to \mathbb{R}$ such that $F' = f$ on $[a,b]$

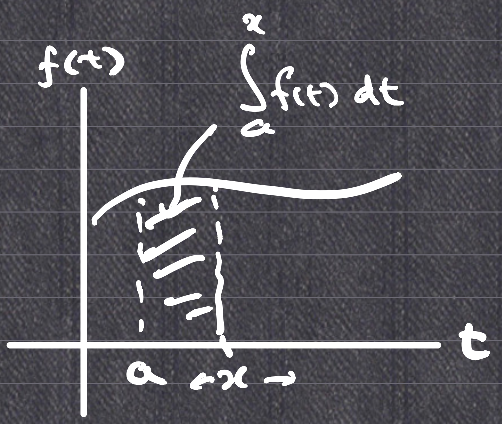
F: primitive function, anti-derivative of f.

then:
$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx = F(b) - F(a) =: \left[F(x)\right]_a^b$$
$$\left(F(x)\Big|_a^b \right)$$

Theorem (Fundamental Theorem of Calculus)

Given f is continuous on $[a,b]$, then

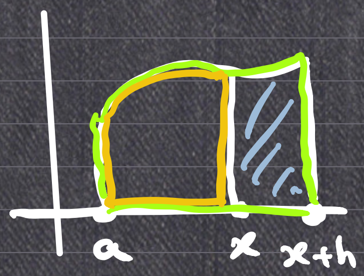
$$\frac{d}{dx} \underbrace{\int_a^x f(t)\,dt}_{\text{function of } x} = f(x).$$



Proof:

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{(x+h)-x}$$



$$= \lim_{h\to 0} f(c(h))$$

$$\uparrow$$

$$\begin{cases} c(h) \in [x, x+h] \\ \qquad \text{or } [x+h, x]. \end{cases}$$

Last time:

$g : [a,b] \to \mathbb{R}$ continuous

then $\exists\, c \in [a,b]$

s.t. $g(c) = \frac{1}{b-a}\int_a^b g(x)\,dx$

$$x \le c(h) \le x+h$$

$$\begin{array}{ccc} x & c(h) & x+h \\ \downarrow & \Downarrow & \downarrow{\scriptstyle h\to 0} \\ x & x & x \end{array}$$

f is continuous $\Rightarrow \lim_{h\to 0} f(c(h)) = f(x)$. ∎

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof of Newton-Leibniz:

Given $\frac{d}{dx}F(x) = f(x)$ and :

by FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$\Rightarrow \frac{d}{dx}\left(\int_a^x f(t)\,dt - F(x)\right) = f(x) - f(x) = 0.$$

$$\Rightarrow \int_a^x f(t)\,dt - F(x) = C \quad \forall x \in [a,b].$$

Put $x = a$: $\overset{0}{\cancel{\int_a^a f(t)\,dt}} - F(a) = C \Rightarrow C = -F(a)$

$$\therefore \quad \int_a^x f(t)\,dt = F(x) + C = F(x) - F(a).$$

$\forall x \in [a, b]$.

In particular:

put $x = b \Rightarrow$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

$\blacksquare$.

$$\int_a^b f(☺)\,d☺ = F(b) - F(a).$$

e.g. $\displaystyle\int_0^1 x^p\,dx = \left[\frac{1}{p+1}x^{p+1}\right]_0^1 = \frac{1}{p+1}$ $\boxed{p \geq 0.}$

$$\frac{d}{dx}\underbrace{\left(\frac{1}{p+1}x^{p+1}\right)}_{F} = \underbrace{x^p}_{f}$$

e.g. $\displaystyle\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = 2.$

$$\frac{d}{dx}(-\cos x) = \sin x$$

e.g. $\displaystyle\int_0^\pi \sin(2x+1)\,dx = \left[-\frac{1}{2}\cos(2x+1)\right]_0^\pi = \cdots$

$$\frac{d}{dx}\underbrace{\left(-\frac{1}{2}\cos(2x+1)\right)} = \frac{1}{2}\cdot 2\sin(2x+1) = \sin(2x+1)$$

$$\cancel{\int_{-1}^1 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^1 = -2}$$

not continuous at $x = 0$.

e.g. f continuous on $\mathbb{R}$.

- $$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{d(x^2)}\int_0^{x^2} f(t)\,dt \cdot \frac{d(x^2)}{dx}$$

  $$= f(x^2)\cdot 2x$$

$$\frac{d}{dx}\int_a^{x} \boxed{f(t)\,dt}$$ exactly x / indep. of x / constant.

- $$\frac{d}{dx}\int_0^{x} x f(t)\,dt = \frac{d}{dx}\Big(x \underbrace{\int_0^{x} f(t)\,dt}_{\text{:+)}}\Big)$$

  $$= \frac{dx}{dx}\cdot\int_0^x f(t)\,dt + x\,\frac{d}{dx}\int_0^x f(t)\,dt$$

  $$= \int_0^x f(t)\,dt + x f(x).$$

- $$\frac{d}{dx}\int_x^{x^2} f(t)\,dt = \frac{d}{dx}\left(\int_0^{x^2} f(t)\,dt - \int_0^{x} f(t)\,dt\right)$$

  $$= 2x f(x^2) - f(x).$$

Prop Given $f:[a,b]\to[0,\infty)$ continuous. ($f \geq 0$)

and $\int_a^b f(x)\,dx = 0$,

then $f(x)=0 \ \forall x\in[a,b]$.

Proof: $F(x) := \int_a^x f(t)\,dt$

FTC $\Rightarrow \frac{d}{dx}F(x) = f(x) \geq 0 \implies F(x)$ is increasing on $[a,b]$.

f is cts

$F(a) \to$ a ———— b $\leftarrow F(b) = \int_a^b f(t)\,dt \boxed{=0}$

$= \int_a^a f(t)\,dt \boxed{=0}$ constant given

$\Rightarrow F(x) = F(a) \ \forall x\in[a,b] \Rightarrow f(x) = \frac{d}{dx}F(x) = 0 \ \forall x\in[a,b]$

$$\underbrace{\int f(x)\,dx}_{\text{indefinite integral}} = \left\{ F : \frac{d}{dx}F = f \right\} = \left\{ F_0 + C : C \in \mathbb{R} \right\}.$$

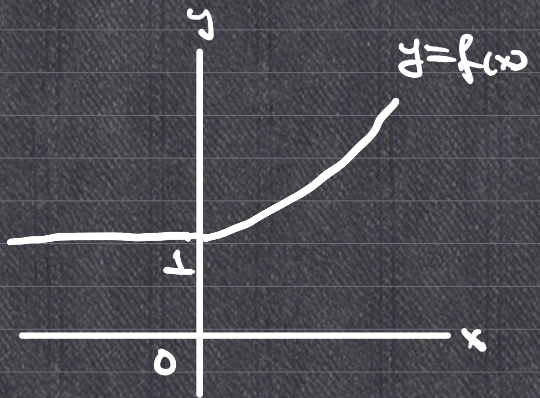
If $F_0' = f$, then $\dfrac{dF}{dx} = f$

$\Rightarrow F = F_0 + C.$

one of the anti-derivatives of f

$$\underbrace{\int f(x)\,dx}_{\left(\frac{d}{dx}\right)^{-1} f(x)} = F_0 + C \quad \text{(where } C \text{ is any real constant)}$$

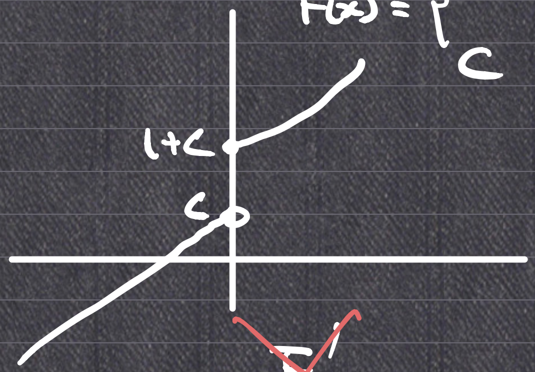
e.g. $f(x) = \begin{cases} e^x & \text{if } x \ge 0 \\ 1 & \text{if } x < 0 \end{cases}$



$$\cancel{\int f(x)\,dx = \begin{cases} e^x + C & \text{if } x \ge 0 \\ x + C & \text{if } x < 0 \end{cases}}$$

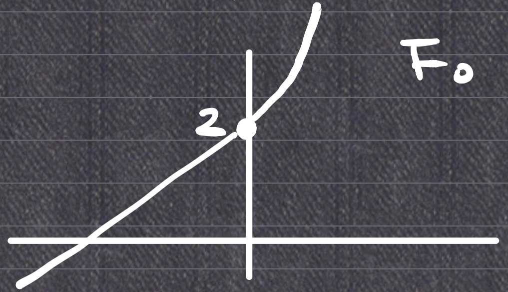
$$\cancel{\int f(x)\,dx = \begin{cases} e^x + C_1 & \text{if } x \ge 0 \\ x + C_2 & \text{if } x < 0 \end{cases}}$$

$F(x) = \begin{cases} e^x + C & \text{if } x \ge 0 \\ C & \text{if } x < 0 \end{cases}$



$\cancel{F'}$

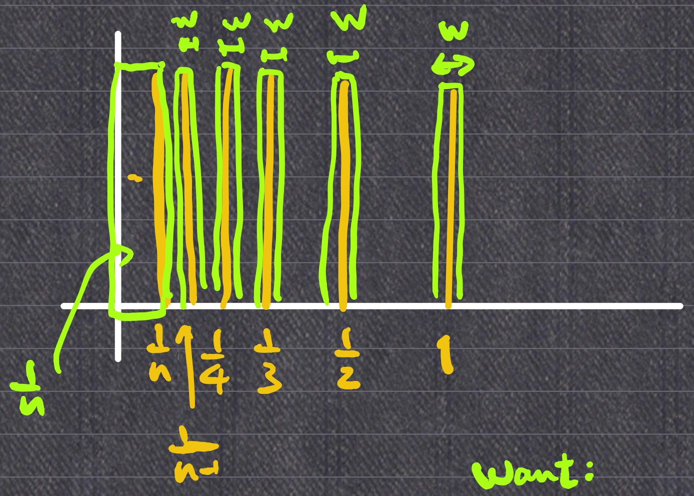
$$F_0(x) = \begin{cases} e^x + 1 & \text{if } x \ge 0 \\ x + 2 & \text{if } x < 0. \end{cases}$$



Ex: $F_0' = f$ on $\mathbb{R}$.

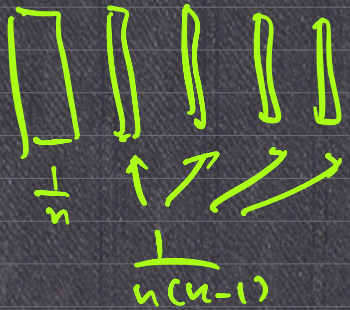
any real const.

$$\therefore \int f(x)\,dx = \begin{cases} e^x + 1 + C & \text{if } x \ge 0 \\ x + 2 + C & \text{if } x < 0 \end{cases}$$



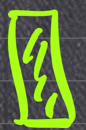
Want: $w(n-1) \le \frac{1}{n}$.

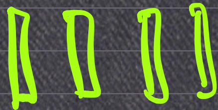
$w \le \frac{1}{n(n-1)}$.

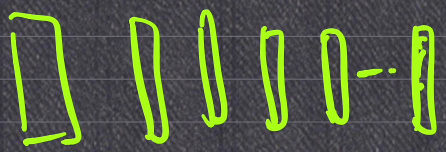
$T_n =$ 

$\underline{A(T_n) = \frac{2}{n} \to 0.}$

$\Rightarrow \quad 0 \le \mu^* \le A(T_n)$

$\underline{\forall \varepsilon > 0}$: $\exists n$ s.t.  $< \frac{\varepsilon}{2}$. and ~~…~~ $\exists\, w > 0$

s.t.  $< \frac{\varepsilon}{2}$

$\Rightarrow$  $< \varepsilon$.

$\Rightarrow \quad 0 \le \underline{\underline{\mu^*}} \le \varepsilon. \quad \forall \varepsilon > 0.$

$\Rightarrow \quad \mu^* = 0.$

$$\underbrace{\mu_*(\bigcirc_n)}_{\pi r^2} \le \mu_*(\circ) \le \mu^*(\circ) := \inf\{A(T) : T \supset \circ\} \le \underbrace{\mu^*(\bigcirc_n)}_{\pi r^2}.$$

$\inf X$

$$\begin{cases} m \text{ is a lower bound for } X \\ \exists\, x_n \in X \text{ s.t. } x_n \to m \end{cases}$$

$\Rightarrow \inf X = m.$

$$\mu^*(\Omega) = \inf \{ A(T) : T \supset \Omega \} = \ ?$$

$\quad$ ↑ simple

$$\begin{cases} \pi r^2 \text{ is lower bound for } \{A(T) : T \supset \Omega\} \\ \text{and } \exists\, T_n \text{ s.t. } A(T_n) \to \pi r^2. \end{cases}$$

$\Rightarrow \inf \{ A(T) : T \supset \Omega \} = \pi r^2.$