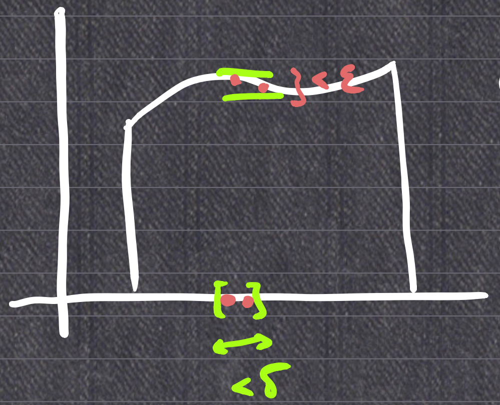
Prop: Any continuous function $f : \underbrace{[a,b]}_{\substack{\text{closed}\\ \text{bounded}}} \to \mathbb{R}$ must be Riemann integrable on $[a,b]$.

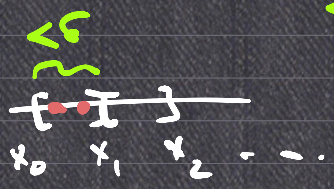
Proof: Recall that such f must be uniformly continuous on $[a,b]$.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $x, y \in [a,b]$ and $|x-y| < \delta$

$$\Rightarrow |f(x) - f(y)| < \boxed{\frac{\varepsilon}{2(b-a)}}$$



Choose P : $a = \underbrace{x_0 < x_1}_{<\delta} < \underbrace{\cdots}_{<\delta} < \underbrace{\cdots < x_n}_{<\delta} = b$.



$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \text{ if } x, y \in \underbrace{[x_{i-1}, x_i]}_{<\delta}.$$

$$\Rightarrow \sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \leq \frac{\varepsilon}{2(b-a)}$$

$$0 \leq U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_n]} f - \inf_{[x_{i-1}, x_i]} f \right)(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{2(b-a)} (x_i - x_{i-1})$$

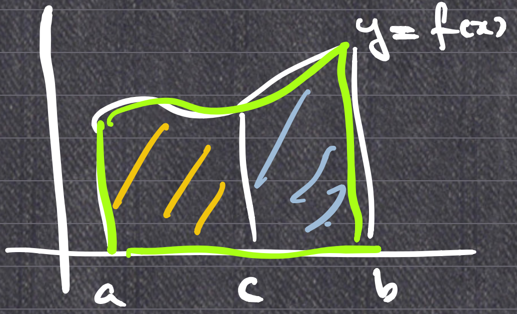
$$= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2} < \varepsilon.$$

$\blacksquare$

FYI: Lebesgue Theorem

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ $\Longleftrightarrow$ $S := \{x_0 \in [a,b] : f \text{ is not continuous at } x_0\}$ has 1-dim Lebesgue measure $= 0$.

① $\boxed{\int_a^b f(x)\,dx} = \boxed{\int_a^c f(x)\,dx} + \boxed{\int_c^b f(x)\,dx}$ directly follows from finite additivity of Jordan measure



② $$\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$$

f is Riemann integrable on $[a,b]$

$\Rightarrow \exists P_n$ s.t. $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$

$= \int_a^b f(x)\,dx.$

$$U(cf,P_n) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} cf\right)(x_i - x_{i-1})$$

$$= \begin{cases} c\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1}) & \text{if } c \geq 0 \\ c\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \cdot (x_i - x_i) & \text{if } c < 0 \end{cases}$$

$$= \begin{cases} c\,U(f,P_n) & \text{if } c \geq 0 \\ c\,L(f,P_n) & \text{if } c < 0 \end{cases}$$

$$\boxed{\lim_{n\to\infty} U(cf,P_n)} = \boxed{\begin{cases} c\int_a^b f(x)\,dx & \text{if } c \geq 0 \\ c\int_a^b f(x)\,dx & \text{if } c < 0 \end{cases}}$$

$$L(cf,P_n) = \begin{cases} c\,L(f,P_n) & \text{if } c \geq 0 \\ c\,U(f,P_n) & \text{if } c < 0. \end{cases}$$

$$\boxed{\lim_{n\to\infty} L(cf,P_n)} = \boxed{c\int_a^b f(x)\,dx.}$$

④ $f(x)$, $g(x)$ Riemann integrable on $[a,b]$

and $f(x) \le g(x)$ on $[a,b]$.

then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

[Because

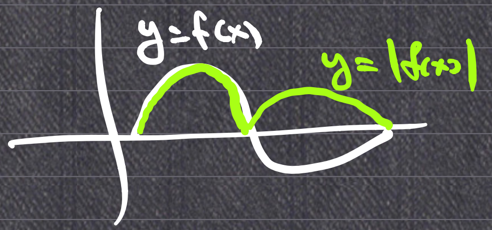
$U(f,P) \le U(g,P)$.]

⑤ If f is Riemann integrable on $[a,b]$

then $|f|$ is also Riemann integrable on $[a,b]$.

and:

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx. \leftarrow$$



Proof:

$$-|f| \le f \le |f|$$

$$-\underline{\int_a^b |f(x)|\,dx} \le \int_a^b f(x)\,dx \le \underline{\int_a^b |f(x)|\,dx}$$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \le \underline{\int_a^b |f(x)|\,dx}$$

☐

③ Given f and g are Riemann integrable on $[a,b]$

then $f+g$ is also Riemann integrable on $[a,b]$

and $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Proof: f is Riemann integrable on $[a,b]$

$\Rightarrow \exists\, P_n$ of $[a,b]$ s.t. $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$

$= \int_a^b f(x)\,dx$.

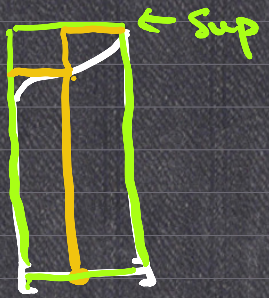
g ... — ... $\Rightarrow \exists\, Q_n$ of $[a,b]$ s.t.

$$\lim_{n\to\infty} U(g,Q_n) = \lim_{n\to\infty} L(g,Q_n) = \int_a^b g(x)\,dx.$$

WANT: find $\{R_n\}$ s.t. $\lim_{n\to\infty} U(f+g,R_n) = \lim_{n\to\infty} L(f+g,R_n)$

Choice: $R_n = P_n \cup Q_n$.

$\rightarrow L(f,P_n) \le L(f,R_n) \le U(f,R_n) \le U(f,P_n)$



$\rightarrow L(g,Q_n) \le L(g,R_n) \le U(g,R_n) \le U(g,Q_n)$.

$$\rightarrow \underset{\{x_i\}_{i=1}^n}{U(f+g,R_n)} = \sum_{i=1}^n \underbrace{\sup_{[x_{i-1},x_i]} (f+g)}\cdot(x_i - x_{i-1})$$

$$\le \sum_{i=1}^n \left(\sup_{[x_{i-1},x_i]} f + \sup_{[x_{i-1},x_i]} g \right)(x_i - x_{i-1})$$

$$= U(f,R_n) + U(g,R_n).$$

$\rightarrow L(f+g,R_n) \ge L(f,R_n) + L(g,R_n)$

$$0 \le U(f+g,R_n) - L(f+g,R_n)$$

$$\le U(f,R_n) + U(g,R_n) - L(f,R_n) - L(g,R_n)$$

$$\le U(f,P_n) + U(g,Q_n) - L(f,P_n) - L(g,Q_n) \longrightarrow 0$$

as $n\to\infty$.

$\therefore$ $f+g$ is Riemann integrable on $[a,b]$.

$$\underbrace{L(f,P_n)}_{\to \int_a^b f(x)dx} + \underbrace{L(g,Q_n)}_{\to \int_a^b g(x)dx}$$

$$\leq L(f,P_n) + L(g,R_n)$$

$$\leq L(f+g,R_n)$$

$$\leq \underline{\int_a^b}(f+g)(x)\,dx = \int_a^b (f+g)(x)\,dx$$

$$= \overline{\int_a^b} f(x)+g(x)\,dx$$

$$\leq U(f+g, R_n) \leq U(f,R_n) + U(g,R_n)$$

$$\leq \underbrace{U(f,P_n)}_{\to \int_a^b f(x)dx} + \underbrace{U(g,Q_n)}_{\to \int_a^b g(x)dx}$$

$$\Longrightarrow \int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \quad \blacksquare$$

$a > b$

$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$$

Prop: $f : [a,b] \to \mathbb{R}$ continuous function.

then $\exists\, c \in [a,b]$ s.t.

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

Proof:

$$M := \sup_{[a,b]} f(x) = f(x_0)$$

$$m := \inf_{[a,b]} f(x) = f(x_1)$$

x_0, x_1 exist by EVT.

$$\underset{\underset{f(x_1)}{\shortparallel}}{m} = \frac{1}{b-a}\int_a^b m\,dx \le \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx} \le \frac{1}{b-a}\int_a^b M\,dx$$

$$m \le \qquad \le M$$

$$= \frac{1}{b-a}\cdot M(b-a)$$

$$= M = f(x_0)$$

By NT: $\exists\, c \in [x_0, x_1]$

s.t. $\quad f(c) = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$

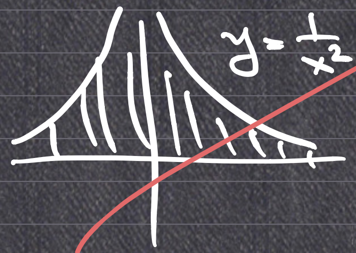
∎

If $\exists F$ s.t. $F'(x) = f(x)$ and $f(x)$ is continuous on $[a,b]$,

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} \qquad \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}.$$

$$= -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2.$$



$$\lim_{n\to\infty} \left(U(f+g, R_n) - L(f+g, R_n) \right) = 0 \quad \checkmark$$

$$\not\Rightarrow \quad \lim_{n\to\infty} U(f+g, R_n) = \lim_{n\to\infty} L(f+g, R_n).$$

$\exists R_{n_k}$ s.t. $\lim U(f+g, R_{n_k})$, $\lim L(f+g, R_{n_k})$ exist.

$$0 = \lim_{k\to\infty} \left(U(f+g, R_{n_k}) - L(f+g, R_{n_k}) \right)$$

$$= \lim_{k\to\infty} U(f+g, R_{n_k}) - \lim_{k\to\infty} L(f+g, R_{n_k}).$$