~~Tor~~ Tutorial 2

Def: Let f be bounded, P be partition

$U(P,f) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

$L(P,f) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

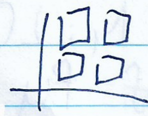
$\overline{\int_a^b} f(x)dx := \inf\{U(P,f) \mid P$ is partition of $[a,b]\}$

$\underline{\int_a^b} f(x)dx := \sup\{L(P,f) : P$ is partition of $[a,b]\}$

f is Riemann Integrable if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$

Tutorial 2

① Explaining relationship between Jordan measure and Riemann integrabilit

- Jordan measure is measuring area of set.
- Riemann integral is measure area between a function and x-axis
- A function has 1 value at each point. Area can be like 

Fact: E is Jordan measurable subset of $[a,b]$

iff $\chi_E : [a,b] \to \mathbb{R}$, defined by $\chi_E(x) = \begin{cases} 1 & \text{when } x \in E \\ 0 & \text{otherwise} \end{cases}$

is Riemann integrable

(Skip the proof)

Application: $\chi_{\mathbb{Q}\cap[0,1]}$ is not Rieman integrable

Since $\mathbb{Q}\cap[0,1]$ is not Jordan measurable

Q2 Let $f : [a,b] \to \mathbb{R}$ be bounded. The following are equivalent

(i) $\forall \varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t. $U(P,f) - L(P,f) < \varepsilon$

(ii) f is Riemann integrable on $[a,b]$

pf: ($\Leftarrow$) Suppose f is Riemann integrable

Let $\varepsilon > 0$, $\exists$ partition P and Q s.t.

$U(f,P) < \overline{\int_a^b} f + \varepsilon$ and $\underline{\int_a^b} f - \varepsilon < L(f,Q)$

Consider $P \cup Q$ (using the hint)

$$\underline{\int_a^b} f - \varepsilon < L(f,Q) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,P) < \overline{\int_a^b} f$$

Since f is Riemann integrable,

$\underline{\int_a^b} f = \overline{\int_a^b} f$

$\Rightarrow U(f, P\cup Q) - L(f, P\cup Q) \leq 2\varepsilon$

($\Rightarrow$) $\forall \varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t. $U(P,f) - L(P,f) < \varepsilon$

Note $L(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P)$

$\Rightarrow 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$

Q2 proof of hint

If P, Q are partition on $[a,b]$ s.t. $P \subseteq Q$

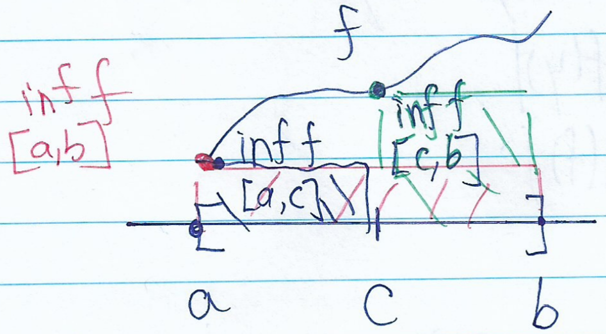
$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

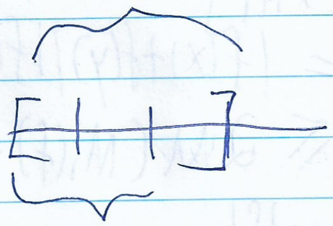
proof by induction on $l := \#Q - \#P$

We can consider $l = 1$

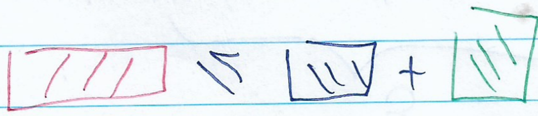
$P = \{a, b\}$

$Q = \{a, c, b\}$ where $a < c < b$





$$\inf_{[a,b]} f \leq \min\left\{\inf_{[a,c]} f, \inf_{[c,b]} f\right\}$$



$$L(f,P) \leq L(f,Q)$$

Fact: $L(-f,P) = -U(f,P)$

$\Rightarrow U(f,Q) \leq U(f,P)$

Q3 (a) Given $\varepsilon > 0$, $\exists$ partition P of $[a,b]$ s.t.

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) < \varepsilon$$

$\forall x, y \in [x_i, x_{i+1}]$

$$|f^2(x) - f^2(y)| = |f(x) + f(y)||f(x) - f(y)| \le 2M(M_i(f) - m_i(f))$$

where $M := \sup_{[a,b]} |f|$

$$\Rightarrow M_i(f^2) - m_i(f^2) \le 2M(M_i(f) - m_i(f))$$

$$U(f^2,P) - L(f^2,P) = \sum_{i=1}^{n} (M_i(f^2) - m_i(f^2))(x_i - x_{i-1})$$

$$\le 2M \underbrace{\sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1})}_{< \varepsilon}$$

$$\le 2M\varepsilon$$ //

(b) $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$

Fact: f, g Riemann integrable $\Rightarrow$ $f+g$, $f-g$ integrable

By (a) $(f+g)^2$, $(f-g)^2$ integrable //

Q4: non-example

$f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 1/p & x = \frac{q}{p} \quad p, q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$g : [0,1] \to \mathbb{R}$

$$g(x) := \begin{cases} 1 & x = \frac{1}{p} \quad p \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$$g \circ f(x) = \begin{cases} 1 & x = \frac{q}{p}, x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

f, g Riemann integrable, $g \circ f$ is not

remark $g \circ f = \chi_{\mathbb{Q} \cap [0,1]}$

$g = \chi_{\{\frac{1}{n}\}_{n=1}^{\infty}}$

~~Pf~~

Let $f : [a,b] \to [c,d]$ be Riemann integrable and $g : [c,d] \to \mathbb{R}$ be continuous. Then $g \circ f$ is continuous

pf: (By continuity of g)

Given $\varepsilon > 0$, $\exists \varepsilon > \delta > 0$ s.t. if $|x-y| < \delta$, $|g(x) - g(y)| < \varepsilon$ if $|x-y| < \delta$

By integrability of f, $\exists P$ s.t. $U(f,P) - L(f,P) < \delta^2$

Let $A := \{ i \mid (M_i - m_i)(f) < \delta \}$

$B := \{ i \mid M_i(f) - m_i(f) \geq \delta \}$

if $i \in A$, $(M_i - m_i)(g \circ f) \leq \varepsilon$ (by continuity of g)

if $i \in B$, $\delta \sum_B (x_i - x_{i-1}) \leq \sum (M_i - m_i)(f)(x_i - x_{i-1}) < \delta^2$

$\Rightarrow \sum (x_i - x_{i-1}) < \delta$

$U(g \circ f, P) - L(g \circ f, P) = \sum_A (M_i - m_i)(g \circ f)(x_i - x_{i-1}) + \sum_B (M_i - m_i)(g \circ f)(x_i$

$$\leq \varepsilon(b-a) + 2\sup|g|\delta$$
$$\leq ((b-a)+2\sup|g|)\varepsilon$$