Tutorial

Q1: Show $\mathbb{Q} \cap [0,1] \times [0,1]$ is not Jordan measurable in $\mathbb{R}$

pf: Let $E := \mathbb{Q} \cap [0,1] \times [0,1]$

$\mu_*(E) = 0$

$\wedge$: $\mu^*(E) = 1$

Suppose not $\mu^*(E) < 1$

$\exists$ simple region $T \subseteq [0,1] \times [0,1]$ s.t $E \subseteq T$

and $\mu^*(E) < A(T) = 1 - \sigma < 1$ for some small $\sigma > 0$

complement of T is simple region too

And it contains recentangle of area larger than 0

That recentangle must contain (q_i, a_i) where $q_i \in \mathbb{Q}$

as $\mathbb{Q}$ is dense in $\mathbb{R}$ //

Q2: Show $\{\frac{1}{n}\}_{n=1}^{\infty} \times [0,1]$ is Jordan measurable

Let $E = \{\frac{1}{n}\}_{n=1}^{\infty} \times [0,1]$

$\forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$ s.t. $\forall m \geq n_1$, $\frac{1}{m} < \varepsilon$

Consider $[0, \varepsilon] \times [0,1]$

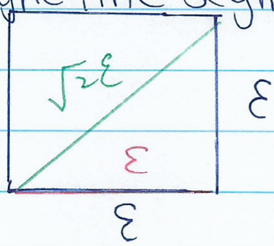
It is a simple region containing $\{\frac{1}{n}\}_{n=n_1}^{\infty} \times [0,1]$

use $\{\frac{1}{n}\}_{n=1}^{n_1 - 1} \times [0,1]$ to cover first n_1 term

total area $= \varepsilon$

Hence $\mu^*(E) = 0 \Rightarrow \mu_*(E) = 0$ and E is Jordan measurable

Q3. Straight line segment has Jordan measure 0



Consider a square of length ε

If a line intersect the square,

at most the length will be $\sqrt{2}\varepsilon$

and at least the length of intersection will be ε

let the length of the line segment be ℓ

we can cover the line segment

by $\frac{\ell}{\varepsilon} + 1$ many square of side length ε

Area $= (\frac{\ell}{\varepsilon} + 1)\varepsilon^2 = \ell\varepsilon + \varepsilon^2$

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Q4: Show Any simple region E in $\mathbb{R}^2$ is Jordan measurable and $\mu(E) = A(E)$

pf: $\mu^*(E) \leq A(E) \leq \mu_*(E) \leq \mu^*(E)$

$\Rightarrow A(E) = \mu_*(E) = \mu^*(E)$ //