Prop (+)

$f : [a,b] \to [0,\infty]$ bounded.

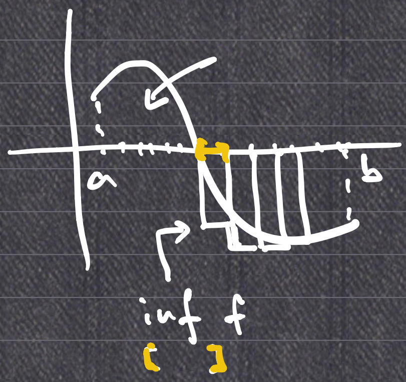
If $\exists \{P_n\}$ a sequence of partitions of $[a,b]$ such that $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = I \in \mathbb{R}$

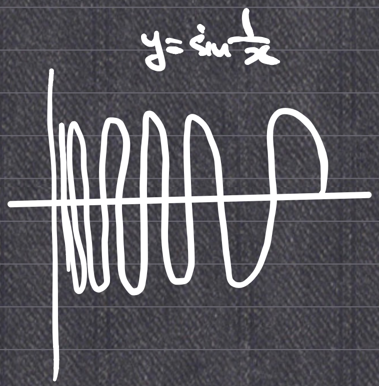
then f is Riemann integrable on $[a,b]$

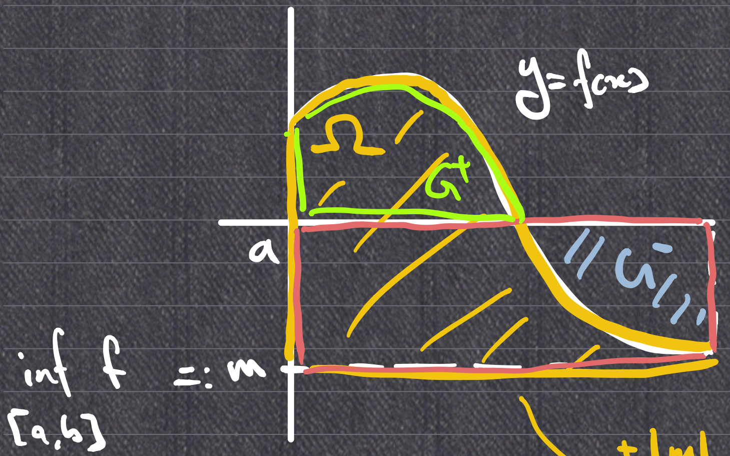
and $\int_a^b f(x)\,dx = I$.

Sketch of proof:

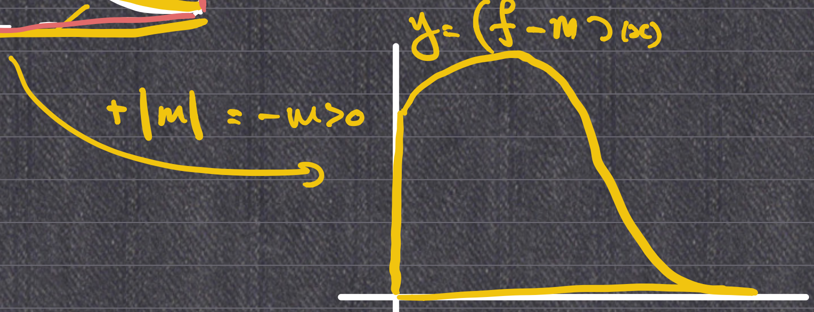
$$L(f,P_n) \le \underbrace{\underline{\int_a^b} f(x)dx}_{=\mu_*(G^+)} \le \underbrace{\overline{\int_a^b} f(x)\,dx}_{=\mu^*(G^+)} \le U(f,P_n).$$







$$\mu(\Omega) + \mu(G^-)$$
$$= \underline{\mu(G^+)} + \underbrace{|m(b-a)|}$$



$$\mu(G^+) - \mu(G^-) = \mu(\Omega) - |m(b-a)|$$
$$= \mu(G^+(\underbrace{f-m}_{\geq 0})) + m(b-a).$$

Prop (R):

$f : [a,b] \to \mathbb{R}$ bounded.

If $\exists \{P_n\}$ a sequence of partitions of $[a,b]$

such that $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = I \in \mathbb{R}$

then f is Riemann integrable on $[a,b]$

and $\int_a^b f(x)\,dx = I$.

Proof:

① Show $f-m$ is Riemann integrable on $[a,b]$

(where $m := \inf_{[a,b]} f(x) < 0$).

$$U(f-m,P) = U(f,P) - m(b-a)$$

[because:

$$\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} (f-m)\cdot(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} \Big(\sup_{[x_{i-1},x_i]} f - m \Big)(x_i - x_{i-1})$$
$$= U(f,P) - \underbrace{\sum_{i=1}^{n} m(x_i - x_{i-1})}_{m(x_n - x_0) = m(b-a)} \ ]$$

Given $\displaystyle\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = I$,

$$\Rightarrow \lim_{n\to\infty} U(\underbrace{f-m}_{\geq 0}, P_n) = \lim_{n\to\infty} L(\underbrace{f-m}_{\geq 0}, P_n) = I - m(b-a).$$

By Prop(+), $f-m$ is Riemann integrable on $[a,b]$.

$\therefore$ f is Riemann integrable on $[a,b]$.

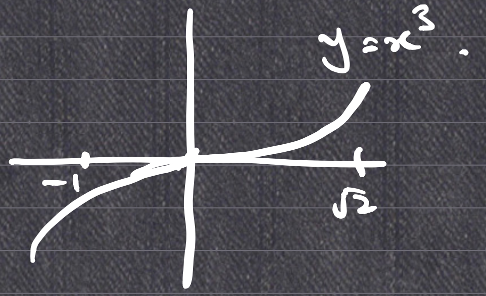
$$\int_a^b f(x)\,dx := \mu(G^+(f)) - \mu(G^-(f)) = \underline{\mu(\Omega)} + m(b-a)$$

$$= \underline{\int_a^b f(x) - m \; dx} + m(b-a)$$

$$= \underline{I - m(b-a)} + m(b-a)$$

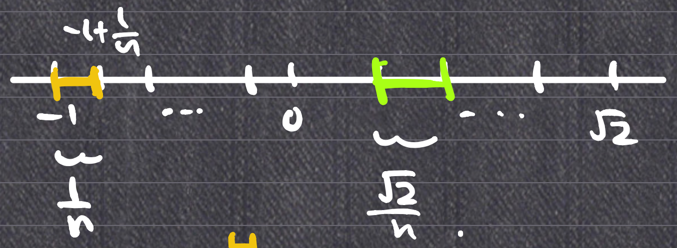
$$= I \quad \blacksquare$$

<u>example</u>: $\displaystyle\int_{-1}^{\sqrt{2}} x^3\,dx$.



let

P_n :



$$\underset{x^3}{U}(f, P_n) = \frac{1}{n}\left(\left(-1+\tfrac{1}{n}\right)^3 + \left(-1+\tfrac{2}{n}\right)^3 + \cdots + \left(-\tfrac{1}{n}\right)^3 + 0^3 \right)$$

$$+ \frac{\sqrt{2}}{n}\left(\left(\frac{\sqrt{2}}{n}\right)^3 + \left(\frac{2\sqrt{2}}{n}\right)^3 + \cdots + \left(\frac{n\sqrt{2}}{n}\right)^3 \right)$$

$$= \frac{1}{n}\left(\frac{(-n+1)^3}{n^3} + \frac{(-n+2)^3}{n} + \cdots + \left(\frac{-n+(n-1)}{n}\right)^3 \right)$$

$$+ \frac{\sqrt{2}}{n}\left(\frac{2^{3/2}}{n^3}\left(1^3+2^3+\cdots+n^3\right) \right)$$

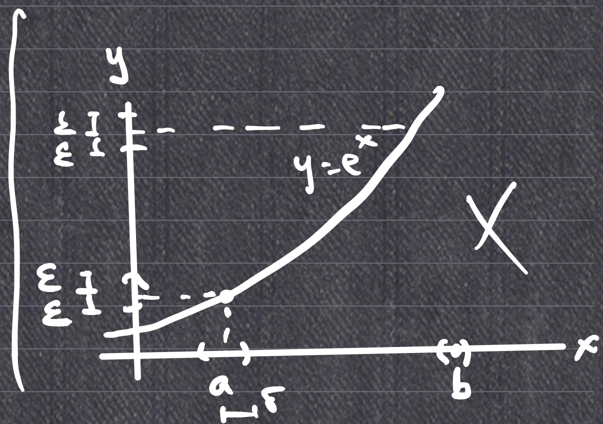
$$\boxed{1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}}$$

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \frac{\sqrt{2}\cdot 2^{3/2}}{4} - \frac{1}{4} = \frac{3}{4}.$$

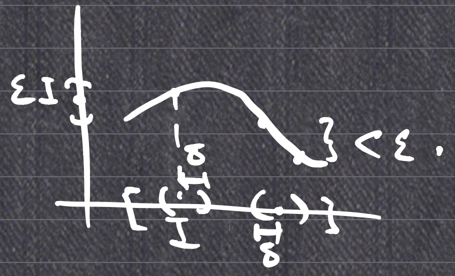
Next goal: Show any continuous function on $[a,b]$ must be Riemann integrable on $[a,b]$.

<u>Uniform continuity</u>

$f: I \to \mathbb{R}$ is uniformly continuous on I $\overset{\text{def}}{\iff}$ $\forall \varepsilon > 0$, $\exists \delta > 0$ <u>indep.</u> of x, y such that



$|x-y|<\delta$ and $x,y\in I$
$\Rightarrow |f(x)-f(y)|<\varepsilon.$



e.g. $f: I\to\mathbb{R}$, $|f'|\le M$ on I.
$\forall \varepsilon>0$, $\exists \delta = \frac{\varepsilon}{M+1}$ indep. of x, y.
$x,y\in I$, $|x-y|<\delta = \frac{\varepsilon}{M+1}$
$\Rightarrow |f(x)-f(y)| = |f'(c)\cdot(x-y)|$, $c\in(x,y)$ or (y,x).
$\le M\underbrace{|x-y|} < M\delta$
$< \frac{\varepsilon M}{M+1} < \varepsilon$

e.g. e^x is not uniformly continuous on $\mathbb{R}$.
but e^x is uniformly continuous on any bounded interval.

Prop: Any continuous function $f:\underbrace{[a,b]}\to\mathbb{R}$ on closed and bounded $[a,b]$ must be uniformly continuous on $[a,b]$.

Proof: Assume not:

NOT: $\left(\forall\varepsilon>0, \exists\delta>0, \text{ s.t. } x,y\in[a,b], |x-y|<\delta \Longrightarrow |f(x)-f(y)|<\varepsilon\right)$

$\Leftrightarrow \left(\exists\varepsilon>0, \forall\delta>0 \text{ s.t. } \exists\, x,y\in[a,b], |x-y|<\delta \text{ but } |f(x)-f(y)|\ge\varepsilon.\right)$

Take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$,

$\exists\, x_n, y_n \in \underset{\substack{\text{closed} \\ \text{bounded.}}}{\underbrace{[a,b]}}$, $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Bolzano-Weierstrass $\Rightarrow$ $\exists\, x_{n_k}, y_{n_k} \in [a,b]$

$$x_{n_k} \xrightarrow{k\to\infty} L, \quad y_{n_k} \xrightarrow{k\to\infty} M.$$

$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \to 0$ as $k \to \infty$ $\Rightarrow$ $L = M$.

__But:__ $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$.

$$\underbrace{\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})|}_{\substack{f \text{ is continuous} \\ \| \\ |f(L) - f(M)| = 0}} \geq \varepsilon \quad \Rightarrow \quad 0 \geq \varepsilon > 0.$$

↯.

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