- Finite additivity:

$\Omega_1$ and $\Omega_2$ are Jordan measureable sets in $\mathbb{R}^2$.

(1) $\Omega_1 \cup \Omega_2$ is also Jordan measurable

(2) If further $\Omega_1 \cap \Omega_2 = \phi$, then

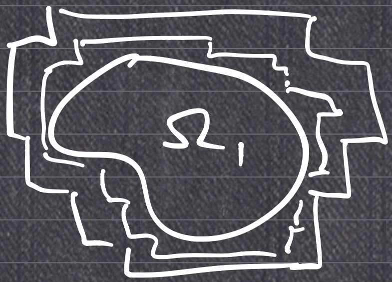
$$\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2).$$

Proof of (1): $\Omega_i$ $(i=1,2)$ is Jordan meas.

$$\Rightarrow \exists\, T_n^i \supset \Omega_i \supset S_n^i \text{ s.t.}$$

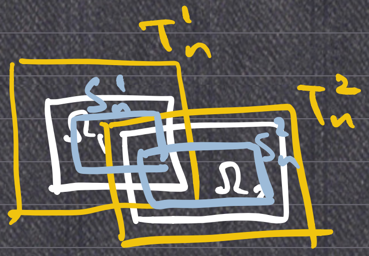
simple

$$\lim_{n\to\infty} A(T_n^i) = \lim_{n\to\infty} A(S_n^i) = \mu(\Omega_i).$$



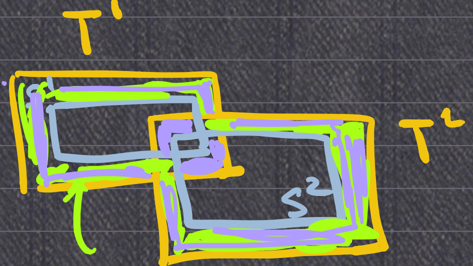

Consider $\underbrace{\{T_n^1 \cup T_n^2\}}, \underbrace{\{S_n^1 \cup S_n^2\}}$

simple regions.




$$T_n^1 \cup T_n^2 \supset \Omega_1 \cup \Omega_2 \supset S_n^1 \cup S_n^2.$$

$$0 \le A(T_n^1 \cup T_n^2) - A(S_n^1 \cup S_n^2) \overset{?}{=} A\left((T_n^1 \cup T_n^2) \setminus (S_n^1 \cup S_n^2)\right)$$




$(T^1 \cup T^2) \setminus (S^1 \cup S^2)$

$$\le A\left((T_n^1 \setminus S_n^1) \cup (T_n^2 \setminus S_n^2)\right)$$

$$\le \underbrace{A(T_n^1 \setminus S_n^1)}_{\to 0} + \underbrace{A(T_n^2 \setminus S_n^2)}_{\to 0}$$

$$(T_n^1 \cup T_n^2) \setminus (S_n^1 \cup S_n^2) \subset (T_n^1 \setminus S_n^1) \cup (T_n^2 \setminus S_n^2)$$

Bolzano-Weierstrass $\Rightarrow$ $\exists$ $A(T_{n_k}^1 \cup T_{n_k}^2)$, converge.

$\lim_{n\to\infty}(b_n - a_n) = 0$. $\quad$ $A(S_{n_k}^1 \cup S_{n_k}^2)$

$a_n \to a_{n_k}$

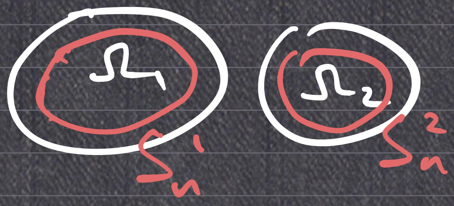
$\therefore$ $\Omega_1 \cup \Omega_2$ is Jordan measurable.

$b_{n_k} \to b_{n_{k_i}}$

$\boxtimes$

(2) If $\Omega_1 \cap \Omega_2 = \phi$, then

$$\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2) \ ?$$




$$\mu(\Omega_1 \cup \Omega_2)$$

$$= \lim_{n\to\infty} A(\underbrace{S_n^1 \cup S_n^2}_{\text{disjoint}})$$

$$= \lim_{n\to\infty} \left(A(S_n^1) + A(S_n^2)\right)$$

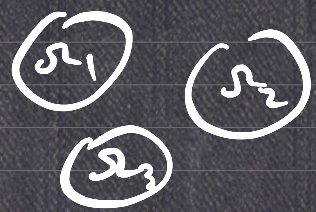
$$= \mu(\Omega_1) + \mu(\Omega_2).$$

$\boxtimes$

By induction, $\Omega_1, \ldots, \Omega_N$ Jordan measurable

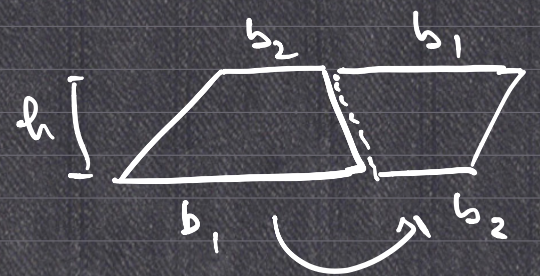
$\Rightarrow$ $\Omega_1 \cup \cdots \cup \Omega_N$ is also Jordan measurable.

Also if $\Omega_i \cap \Omega_j = \phi$ $\forall i \neq j$.




then $\mu(\Omega_1 \cup \cdots \cup \Omega_N)$

$$= \mu(\Omega_1) + \cdots + \mu(\Omega_N).$$




$\mu_1(\Omega_1 \cup \Omega_2)$
$= \mu(\Omega_1) + \mu(\Omega_2) - \mu(\Omega_1 \cap \Omega_2)$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a distance-preserving map

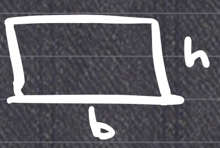
isometry.

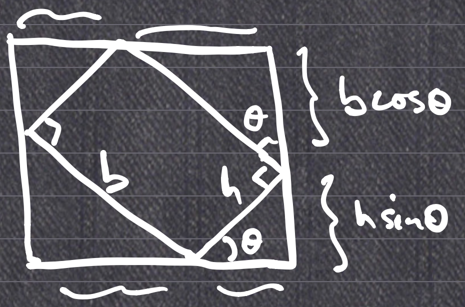
$\left( |\Phi(\vec{x}) - \Phi(\vec{y})| = |\vec{x} - \vec{y}| \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^2 \right)$

Claim: $\Omega$ Jordan measurable in $\mathbb{R}^2$

then $\Phi(\Omega)$ also Jordan measurable

and $\mu(\Phi(\Omega)) = \mu(\Omega)$.

Proof: If $\Omega =$ 


$\mu(X \setminus Y)$

$Y \subset X$

$\mu(X) = \mu(X \cup Y)$
$= \mu((X \setminus Y) \sqcup Y)$
$= \mu(X \setminus Y) + \mu(Y)$


$\mu($  $) = bh$

If $\Omega = \text{simple region} = \coprod_i R_i$

$$\Phi(\Omega) = \coprod_i \Phi(R_i)$$

$$\mu(\Phi(\Omega)) = \sum_i \mu(\Phi(R_i)) = \sum_i \mu(R_i) = \mu\left(\coprod_i R_i\right)$$
$$= \mu(\Omega).$$

General case: $\Omega \subset \mathbb{R}^2$ Jordan measurable.

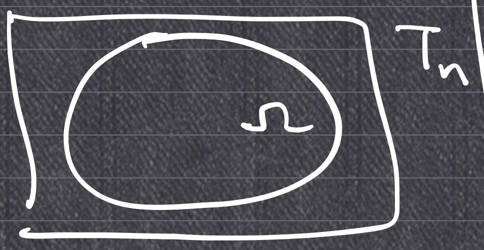
$\Rightarrow \exists\, T_n \supset \Omega \supset S_n$ st. $\lim_{n\to\infty} A(T_n) = \lim_{n\to\infty} A(S_n)$

simple regions $= \mu(\Omega)$.

$$\mu^*(\Phi(\Omega)) \leq \mu^*(\Phi(T_n))$$
$$= \mu(\Phi(T_n))$$
$$= \mu(T_n)$$

because: Given $\boxed{\Omega \subset T_n}$

$\Rightarrow \Phi(\Omega) \subset \Phi(T_n)$.




If $\vec{x} \in \Phi(\Omega)$

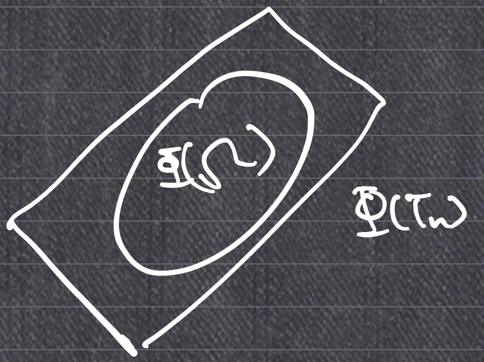
$\Rightarrow \vec{x} = \Phi(\vec{y})$

$\exists\, \vec{y} \in \Omega$

$\Rightarrow \vec{y} \in T_n$

$\Rightarrow \vec{x} = \Phi(\vec{y})$

$\subset \Phi(\Omega)$




$S_n \subset \Omega \Rightarrow \Phi(S_n) \subset \Phi(\Omega)$

$$\mu(S_n) = \mu_*(\Phi(S_n)) \leq \mu_*(\Phi(\Omega))$$

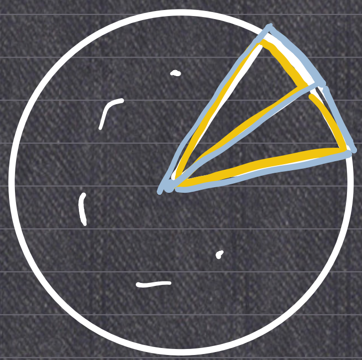
$$\mu(S_n) \leq \mu_*(\Phi(\Omega)) \leq \mu^*(\Phi(\Omega)) \leq \mu(T_n)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\mu(\Omega) \qquad\qquad\qquad\qquad \mu(\Omega)$$

$\therefore\ \Phi(\Omega)$ is Jordan measurable

and $\mu(\Phi(\Omega)) = \mu(\Omega)$.



$$\underbrace{\mu(⬡_n)}_{= \cdots} \le \mu_*(O) \le \mu^*(O) \le \underbrace{\mu(⬡_n)}_{= \cdots}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

$$\lim_{n \to \infty} 3^n \sin \frac{x}{3^n}$$

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$$\underbrace{X}_{\text{bounded.}} \subset Y \subset \mathbb{R}^2$$

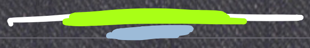
why $\mu^*(X) \le \mu^*(Y)$?

$\mu_*(X) \le \mu_*(Y)$

$X \subset Y$.

$\mu^*(X) = \inf\{A(T) : T \supset X\}$.

$\mu^*(Y) = \inf\{A(T) : T \supset Y\}$.



$$\{A(T) : T \supset Y\} \subset \{A(T) : T \supset X\}$$

$$\mu^*(Y) = \inf\{A(T) : T \supset Y\} \ge \inf\{A(T) : T \supset X\} = \mu^*(X)$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$