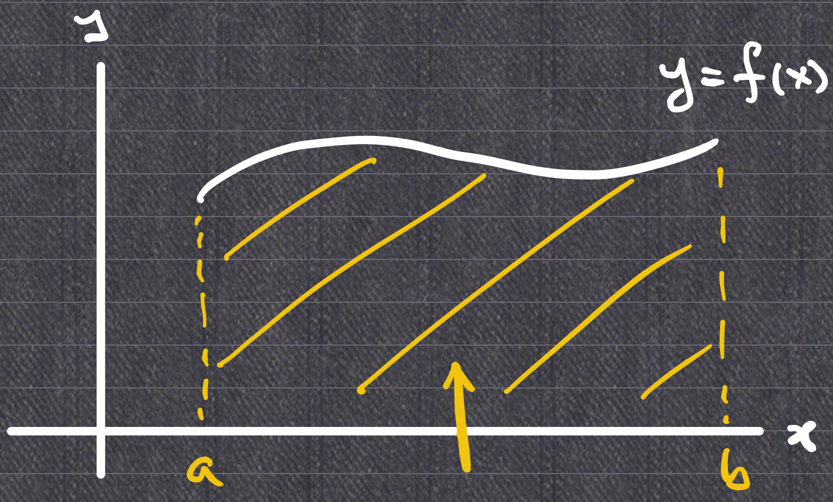
§4.1 Jordan Measure.



$$\text{area} =: \int_a^b f(x)\,dx$$

2. It is known that the shaded regions in Figure I and Figure II are identical, and the area of each square in Figure I and Figure II are 1 cm^2 and 0.25 cm^2 respectively.

Figure I

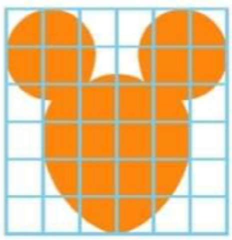


Figure II

(a) Which figure do you think should be used to measure the area of the shaded region more accurately?

(b) According to the figure chosen in **(a)**, measure the area of the shaded region.

Simple regions (in $\mathbb{R}^2$)

A set $\Omega \subset \mathbb{R}^2$ is called a simple region

$\overset{def}{\iff}$ $\Omega = \bigcup_{i=1}^{N} R_i$

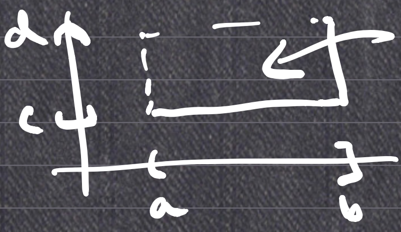
↑ finite

rectangle $= \langle a,b \rangle \times \langle c,d \rangle$

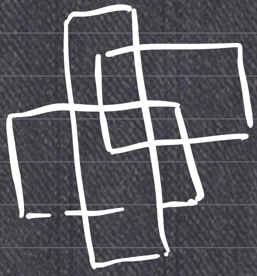
$\langle \; = \; ($ or $[$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

$$A \times B = \{(x,y) : x \in A, \; y \in B\}.$$

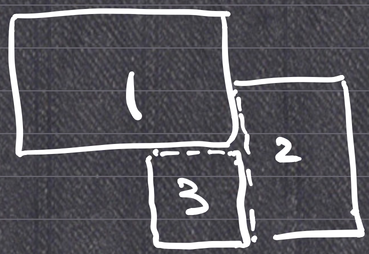


$\{(x,y) : x \in (a,b], \; y \in [c,d)\}$.



<u>Prop:</u> Any simple region Ω in $\mathbb{R}^2$ can be written as a <u>disjoint</u> union of finitely many rectangles.

$$\Omega = \coprod_{i=1}^{N} R_i$$

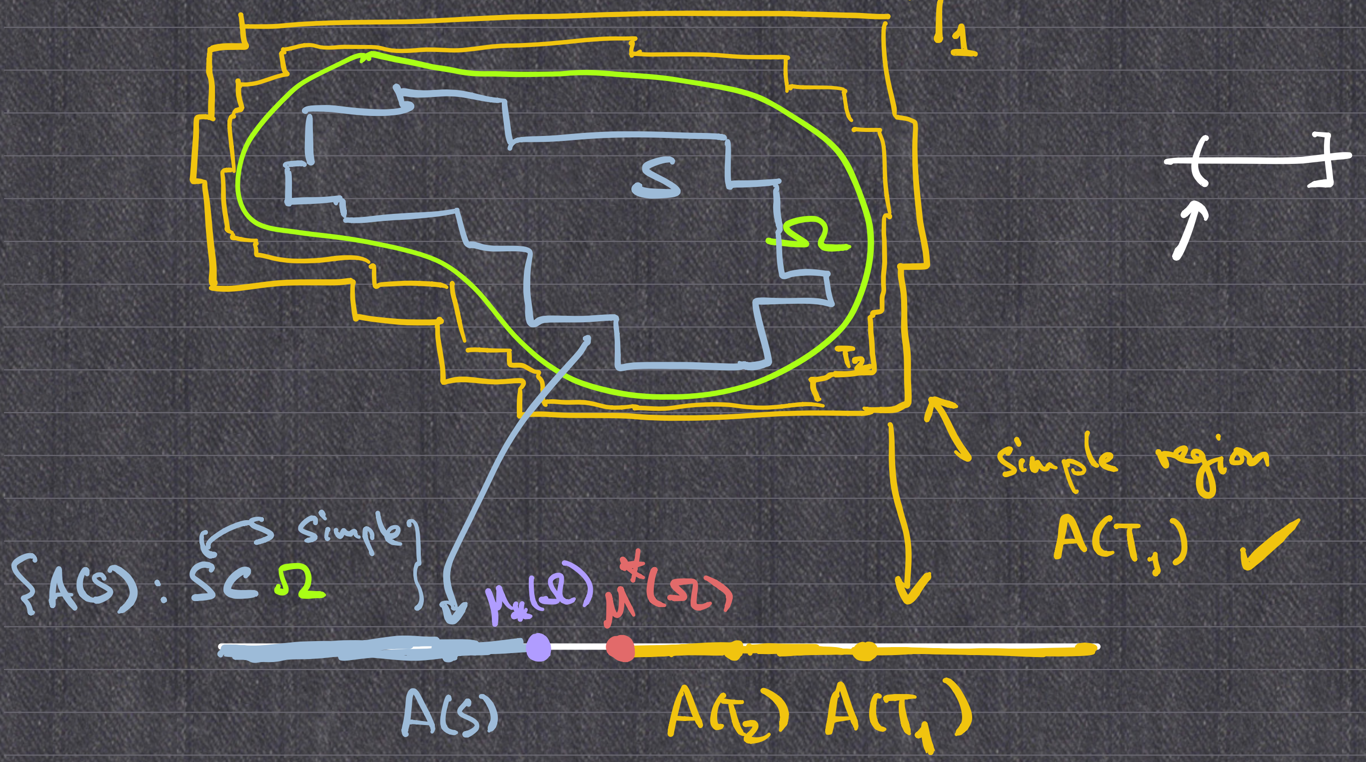


$R_i = \langle a_i, b_i \rangle \times \langle c_i, d_i \rangle$

$A(R_i) := (b_i - a_i) \cdot (d_i - c_i)$

Define $A(\Omega) := \sum_{i=1}^{N} A(R_i) = \sum_{i=1}^{N} (b_i - a_i)(d_i - c_i)$.

General <u>bounded</u> region Ω



$$\mu^*(\Omega) := \inf\{A(T) : T \supset \Omega,\ T \text{ simple}\}.$$

greatest lower bound.

Jordan outer measure

$$\mu_*(\Omega) := \sup\{A(S) : S \subset \Omega,\ S \text{ simple}\}$$

Jordan inner measure.

<u>Def:</u> $\Omega \subset \mathbb{R}^2$ is <u>Jordan measurable</u>

bounded

$\overset{\text{def}}{\iff} \quad \mu_*(\Omega) = \mu^*(\Omega)$

then $\mu(\Omega) := \mu_*(\Omega)$.

$T_0 \supset \Omega \supset S_0$

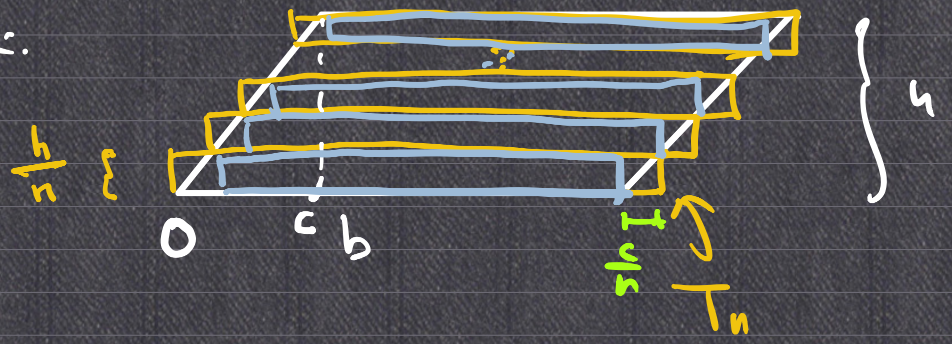
$\uparrow$

Simple

$$A(T_0) \geq \mu^*(\Omega) = \inf\left\{A(T) : \underset{T \text{ simple}}{T \supset \Omega}\right\}.$$

$$A(S_0) \leq \mu_*(\Omega).$$

e.g.



$$\frac{h}{c} = \frac{\frac{h}{n}}{\vdash\!\dashv}$$

$$A(T_n) = n \cdot \frac{h}{n} \cdot \left(b + \frac{c}{n}\right) = h\left(b + \frac{c}{n}\right)$$

$$A(S_n) = n \cdot \frac{h}{n} \cdot \left(b - \frac{c}{n}\right) = h\left(b - \frac{c}{n}\right)$$

$$\forall n : \quad \underset{\substack{\| \\ h\left(b - \frac{c}{n}\right) \\ \downarrow \\ hb}}{A(S_n)} \leq \mu_*(\text{▱}) \leq \mu^*(\text{▱}) \leq \underset{\substack{\| \\ h\left(b + \frac{c}{n}\right) \\ \downarrow \\ hb}}{A(T_n)}$$

as $n \to \infty$

$$\Rightarrow \quad \mu_*(\text{▱}) = \mu^*(\text{▱}) = hb.$$

$\therefore$ ▱ is Jordan measurable

and $\mu(\text{▱}) = hb.$

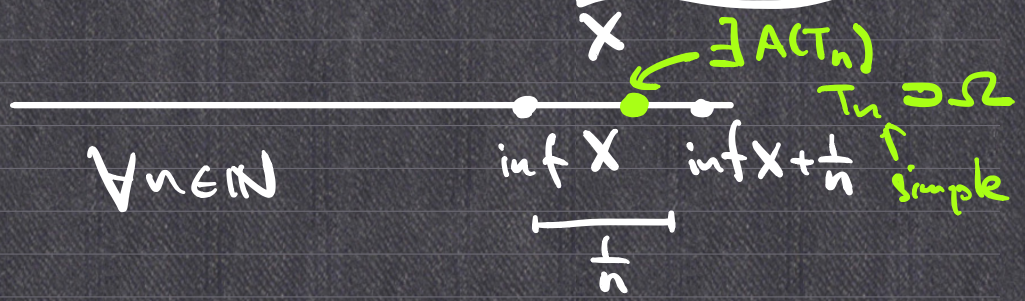
Prop: $\exists\, S_n \subset \Omega \subset T_n$ s.t. $\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty} A(T_n) = m$

simple regions

$\iff$ Ω is Jordan measurable with $\mu(\Omega) = m$.

Proof: ($\Rightarrow$) ✓

($\Leftarrow$) $m = \mu^*(\Omega) = \inf \{ \underbrace{A(T) : \underset{\text{simple}}{T \supset \Omega}}_{X} \}$.



$\forall n: \ \mu^*(\Omega) \leq A(T_n) < \mu^*(\Omega) + \frac{1}{n}$.

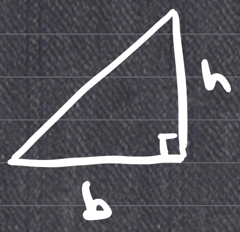
$n \to \infty: \quad \mu^*(\Omega) = \lim_{n \to \infty} A(T_n)$. $\quad \mu^*(\Omega)$

$\mu_*(\Omega)$: proof is similar, mutatis mutandis.

ex:

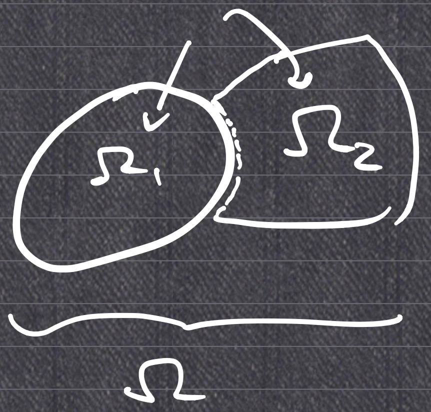
$\mu = 0$. $\qquad 0 \leq \mu_*(\Omega) \leq \mu^*(\Omega)$

ex:



$\frac{1}{2} bh$.

Jordan measurable



$\mu(\Omega) \overset{?}{=} \mu(\Omega_1) + \mu(\Omega_2)$

HW ex: