4.5 Integration by Parts

"Integration by parts" is another important technique of doing integrations. It is a consequence of the product rule.

Proposition 4.18 — Integration by Parts. Let $f, g : [a, b] \to \mathbb{R}$ be two C^1 functions, then

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx.$$

Proof. First recall that

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x).$$

By the Newton-Leibniz's formula (4.3), we have:

$$\int_a^b \big(f'(x)g(x) + g'(x)f(x)\big)\,dx = [f(x)g(x)]_a^b.$$

The desired result follows immediately by rearrangement. ■

ⓘ If we let $u = f(x)$ and $v = g(x)$, then $f'(x)\,dx$ can be regarded as du, and $g'(x)\,dx$ as dv. The integration by parts formula is often expressed as

$$\int_{x=a}^{x=b} u\,dv = [uv]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du.$$

ⓘ With almost the same proof, the integration by parts formula has an indefinite integral version:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Using Proposition 4.18, we can now integrate $\log x$. Letting $f(x) = \log x$ and $g(x) = x$, then on an interval of all positive numbers, we have

$$\begin{aligned}\int \log x\,dx &= x\log x - \int x d(\log x)\\ &= x\log x - \int x\cdot\frac{1}{x}\,dx\\ &= x\log x - \int 1\,dx\\ &= x\log x - x + C.\end{aligned}$$

■ **Exercise 4.38** Using integration by parts, find the integrals:

$$\int_1^2 \frac{\log x}{x^2}\,dx, \qquad \int \tan^{-1} x\,dx, \qquad \int x^3 e^{x^2}\,dx.$$

■ **Example 4.11** Let's integrate $\sec^3 x$ – there is a small trick that is often useful when integrating trigonometric functions.

$$\begin{aligned}\int \sec^3 x\,dx &= \int \sec x \cdot \sec^2\,dx\\ &= \int \sec x d(\tan x)\\ &= \sec x\tan x - \int \tan x d(\sec x)\\ &= \sec x\tan x - \int \tan^2 x \sec x\,dx\\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

Now we see that $\int \sec^3 x\,dx$ appears again but fortunately with a "good" sign in front. By rearrangement, we get

$$2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \log|\sec x + \tan x| + C.$$

We conclude that

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x| + C$$

where C' is any constant.

■ **Exercise 4.39** Compute the integrals:

$$\int e^x \cos x\,dx, \qquad \int \tan^3 x\,dx.$$

4.5.1 Irrationality of e, again

Here we give "another" proof of e being irrational using integrals. For each $n \in \mathbb{N} \cup \{0\}$, we define

$$I_n := \int_0^1 x^n e^{-x}\,dx.$$

First of all, we argue that $0 < I_n < \frac{1}{e}$ for any $n \in \mathbb{N}$ and $x \in [0,1]$. Clearly $I_n \geq 0$. Note that $I_n \neq 0$ since the integrand $x^n e^{-x}$ is non-negative and not identically 0 on $[0,1]$. The function $x^n e^{-x}$ is strictly increasing on $x \in [0,1]$ for any $n \in \mathbb{N}$ (as $(x^n e^{-x})' = x^{n-1}(n-1)e^{-x} \geq 0$). Therefore, we have $x^n e^{-x} \leq 1^n e^{-1} = \frac{1}{e}$ for any $x \in [0,1]$ and $n \in \mathbb{N}$, and so

$$I_n = \int_0^1 x^n e^{-x}\,dx \leq \int_0^1 \frac{1}{e}\,dx = \frac{1}{e}.$$

Next we derive a relation between I_{n+1} and I_n using integration by parts:

$$\begin{aligned}I_{n+1} = \int_0^1 x^{n+1}e^{-x}\,dx &= \int_0^1 x^{n+1} d(-e^{-x})\\ &= [-x^{n+1}e^{-x}]_0^1 - \int_0^1 (-e^{-x})(n+1)x^n\,dx\\ &= -\frac{1}{e} + (n+1)\int_0^1 x^n e^{-x}\,dx = -\frac{1}{e} + (n+1)I_n.\end{aligned}$$

Let $J_n := \frac{I_n}{n!}$ for any $n \in \mathbb{N} \cup \{0\}$, then

$$J_{n+1} = \frac{I_{n+1}}{(n+1)!} = \frac{1}{(n+1)!}\left(-\frac{1}{e} + (n+1)I_n\right) = -\frac{1}{(n+1)!e} + J_n.$$

This shows

$$J_n = J_0 + \sum_{k=1}^{n}(J_k - J_{k-1}) = I_0 - \sum_{k=1}^{n}\frac{1}{k!e} = 1 - \frac{1}{e} - \sum_{k=1}^{n}\frac{1}{k!e} = 1 - \frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}.$$

Therefore, for any $n \in \mathbb{N} \cup \{0\}$,

$$I_n = n!\left(1 - \frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right) = \frac{n!}{e}\left(e - \sum_{k=0}^{n}\frac{1}{k!}\right)$$

Assume that e is rational, then there exist $p, q \in \mathbb{N}$ such that $e = \frac{p}{q}$. Recall that $0 < I_n < \frac{1}{e}$ for any $n \geq 1$, and so

$$0 < n!\left(e - \sum_{k=0}^{n}\frac{1}{k!}\right) < 1.$$

Take $n > q$, then $n!e = n!\frac{p}{q} \in \mathbb{N}$. Clearly, $n!\sum_{k=0}^{n}\frac{1}{k!} = \sum_{k=0}^{n}\frac{n!}{k!} \in \mathbb{N}$ too, so

$$n!\left(e - \sum_{k=0}^{n}\frac{1}{k!}\right) \in \mathbb{Z},$$

but it is clearly absurd as $(0, 1) \cap \mathbb{Z} = \emptyset$. This shows e is irrational.

We put "another" proof in quote because it is not really a new proof from what we have seen in MATH 1023. The integral I_n actually came from the remainder of the Taylor's series of e^x. We will discuss more in Proposition 4.19.

4.5.2 Reduction formulae

In the above proof of the irrationality of e, we derived a recurrence relation for I_n using integration by parts. It is also a very common technique for evaluating complicated integrals.

■ **Example 4.12** For any $m, n \in \mathbb{N} \cup \{0\}$, we define

$$I_{m,n} := \int \cos^m x \sin^n x \, dx.$$

Show that

$$I_{m,n} = -\frac{1}{m+n}\cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+n} I_{m,n-2}, \quad \forall m \geq 0, n \geq 2 \tag{4.4}$$

$$I_{m,n} = \frac{1}{m+n}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}, \quad \forall m \geq 2, n \geq 0 \tag{4.5}$$

Solution We just prove (4.4) and leave (4.5) as an exercise.

$$
\begin{aligned}
I_{m,n} &= \int \cos^m x \sin^n x \, dx = \int \cos^m x \sin^{n-1} x d(-\cos x) \\
&= -\cos^{m+1} x \sin^{n-1} x + \int \cos x \, d(\cos^m x \sin^{n-1} x) \\
&= -\cos^{m+1} x \sin^{n-1} x \\
&\quad + \int \cos x \big(m \cos^{m-1} x(-\sin x) \sin^{n-1} x + \cos^m x \cdot (n-1) \sin^{n-2} x \cdot \cos x\big) \, dx \\
&= -\cos^{m+1} x \sin^{n-1} x - m I_{m,n} + (n-1) \int \cos^{m+2} x \sin^{n-2} x \, dx \\
&= -\cos^{m+1} x \sin^{n-1} x - m I_{m,n} + (n-1) \int \cos^m x (1 - \sin^2 x) \sin^{n-2} x \, dx \\
&= -\cos^{m+1} x \sin^{n-1} x - m I_{m,n} + (n-1) I_{m,n-2} + (n-1) I_{m,n}.
\end{aligned}
$$

By rearrangement, we get (4.4).

Exercise 4.40 Prove (4.5).

Using (4.4), one can then compute some complicated integrals such as

$$\int \cos^4 x \sin^6 x \, dx = I_{4,6} = -\frac{1}{4+6} \cos^5 x \sin^4 x + \frac{6-1}{4+6} I_{4,4}$$

By applying (4.4) again on $I_{4,4}$, we can reduce it to $I_{4,2}$; and apply (4.4) again we get $I_{4,0}$.

Next we apply (4.5) on $I_{4,0}$ and reduce it to $I_{2,0}$, which can be easily computed by half-angle formula:

$$I_{2,0} = \int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{x} + \frac{1}{4} \sin 2x + C.$$

Exercise 4.41 Complete the above reduction procedure and find the full expression of

$$\int \cos^4 x \sin^6 x \, dx.$$

Exercise 4.42 For $I_{m,n}$ when at least one of m and n is odd, we could just use one of the (4.4) and (4.5). Explain why?

Let's also see an example of definite integrals

Example 4.13 For any $n \in \mathbb{N} \cup \{0\}$, let

$$I_n := \int_0^1 x^n \sqrt{1-x} \, dx.$$

Find a recurrence relation for $\{I_n\}$, and deduce its general term.

■ **Solution** Using integration by parts, we can prove that for any $n \geq 1$:

$$\begin{aligned}
I_n &= -\frac{2}{3}\int_0^1 x^n\, d((1-x)^{3/2}) \\
&= -\frac{2}{3}\left[x^n(1-x)^{3/2}\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)^{3/2}\cdot nx^{n-1}\,dx \\
&= 0 + \frac{2n}{3}\int_0^1 (1-x)\cdot x^{n-1}\sqrt{1-x}\,dx \\
&= \frac{2n}{3}\int_0^1 x^{n-1}\sqrt{1-x}\,dx - \frac{2n}{3}\int_0^1 x^n\sqrt{1-x}\,dx \\
&= \frac{2n}{3}I_{n-1} - \frac{2n}{3}I_n.
\end{aligned}$$

By rearrangement, we get:

$$I_n = \frac{2n}{2n+3}I_{n-1} \quad \forall n \geq 1.$$

One can then apply this recurrence relation inductively and get for any $n \in \mathbb{N}$:

$$I_n = \frac{2n}{2n+3}I_{n-1} = \frac{2n}{2n+3}\cdot\frac{2n-2}{2n+1}I_{n-2} = \cdots = \frac{(2n)(2n-2)(2n-4)\cdots(4)(2)}{(2n+3)(2n+1)(2n-1)\cdots(7)(5)}I_0.$$

We can compute that

$$I_0 = \int_0^1 \sqrt{1-x}\,dx = \left[-\frac{2}{3}(1-x)^{3/2}\right]_0^1 = \frac{2}{3}.$$

This concludes that for any $n \in \mathbb{N}$:

$$I_n = 2\times\frac{(2n)!!}{(2n+3)!!}.$$

■ **Exercise 4.43** For any $m, n \in \mathbb{N}\cup\{0\}$, we define:

$$I_{m,n} := \int_0^\pi e^{mx}\sin^n x\,dx.$$

Find a recurrence relation between $I_{m,n}$ and $I_{m,n-2}$, and show that:

$$I_{m,n} = \frac{n!(e^{m\pi}-1)}{m(m^2+4)(m^2+16)\cdots(m^2+n^2)}$$

for any $m \in \mathbb{N}\cup\{0\}$ and **even** $n \in \mathbb{N}$.

■ **Exercise 4.44 — Source: HKAL 1996 Paper II Q12.** For non-negative integers k and m, define

$$F(k,m) = \int_0^1 u^k(1-u^2)^m\,du.$$

(a) Show that

$$\begin{aligned}
F(k,0) &= \frac{1}{k+1} \\
F(k,m) &= \frac{2m}{k+1}F(k+2,m-1) \quad \text{for } m \geq 1.
\end{aligned}$$

(b) Show that
$$F(k,m) = \frac{2^m(m!)}{(k+1)(k+3)\cdots(k+2m+1)}.$$

(c) Using (b), prove that $\displaystyle\int_0^{\pi/2} \cos^{2m+1}\theta\, d\theta = \frac{\left(2^m(m!)\right)^2}{(2m+1)!}$.

(d) Show that $\displaystyle F(k,m) = \sum_{r=0}^{m} \frac{(-1)^r C_r^m}{2r+k+1}$.

■ **Exercise 4.45 — Source: HKAL 2010 Paper II Q9 (restructured).** Answer the following questions:

(a) Prove that $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

(b) For any positive integer n, define $\displaystyle I_n := \int_1^e x^{-3}(\log x)^n\, dx$. Prove that for any $n \in \mathbb{N}$:
$$I_n = n!\left(\frac{1}{2^{n+1}} - \frac{1}{e^2}\sum_{k=0}^{n} \frac{1}{(n-k)!2^{k+1}}\right).$$

(c) Prove that $e^{-2}x^{-1}(\log x)^n \le x^{-3}(\log x)^n \le x^{-1}(\log x)^n$ for all $x \in [1,e]$. Hence prove that
$$\frac{1}{e^2(n+1)} \le I_n \le \frac{1}{n+1}.$$

(d) Using the above results, evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}$.

4.5.3 Taylor's remainder in integral form

Using integration by parts, one can derive a new form of remainder to the Taylor series (in addition to the Cauchy's and Lagrange's forms discussed in MATH 1023).

Proposition 4.19 — Taylor's Remainder in Integral Form. Suppose f is C^{n+1} on an interval I containing a, then we have:
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt$$
for any $x \in I$.

Proof. The key idea is to use integration by parts repeatedly. For each $m, n \in \mathbb{N}$ and $m \ge n$, we have:
$$\begin{aligned}
I_{m,n} &:= \int_a^x (x-t)^m f^{(n+1)}(t)\, dt \\
&= \int_a^x (x-t)^m d\big(f^{(n)}(t)\big) \\
&= \left[(x-t)^m f^{(n)}(t)\right]_{t=a}^{t=x} - \int_a^x f^{(n)}(t)\cdot m(x-t)^{m-1}\cdot(-1)\, dt \\
&= -(x-a)^m f^{(n)}(a) + mI_{m-1,n-1}.
\end{aligned}$$

Apply this recurrence relation repeatedly, we get

$$\begin{aligned}
I_{m,n} &= -(x-a)^m f^{(n)}(a) + m\left(-(x-a)^{m-1} f^{(n-1)}(a) + (m-1)I_{m-2,n-2}\right) \\
&= -(x-a)^m f^{(n)}(a) - m(x-a)^{m-1} f^{(n-1)}(a) \\
&\quad + m(m-1)\left(-(x-a)^{m-2} f^{(n-2)}(a) + (m-2)I_{m-3,n-3}\right) \\
&= \cdots \\
&= -(x-a)^m f^{(n)}(a) - m(x-a)^{m-1} f^{(n-1)}(a) \\
&\quad - m(m-1)(x-a)^{m-2} f^{(n-2)}(a) - m(m-1)(m-2)(x-a)^{m-3} f^{(n-3)}(a) \\
&\quad - \cdots - m(m-1)(m-2)\cdots(m-n+2)(x-a)^{m-n+1} f'(a) \\
&\quad + m(m-1)(m-2)\cdots(m-n+1)I_{m-n,0} \\
&= \frac{m!}{(m-n)!} I_{m-n,0} - \sum_{k=1}^{n} \frac{m!}{(m-n+k)!}(x-a)^{m-n+k} f^{(k)}(a)
\end{aligned}$$

In particular, when $m = n$, we have:

$$\begin{aligned}
\frac{1}{n!} I_{n,n} &= \int_a^x f'(t)\,dt - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(x) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.
\end{aligned}$$

By rearrangement, we can see that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

as desired. ■

Combining Proposition 4.19 with Exercise 4.7, one can give another proof of the Cauchy's remainder theorem. Proposition 4.19 asserts that the remainder of $R_n(x) = f(x) - T_n(x)$ is given by:

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt,$$

and Exercise (4.7) shows that exists c between a and x such that

$$\int_a^x (x-t)^n f^{(n+1)}(t)\,dt = (x-a)\cdot(x-c)^n f^{(n+1)}(c).$$

It shows

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

which is exactly the Cauchy's remainder.

■ **Example 4.14** Prove that for any $n \in \mathbb{N}$, we have

$$\left|\left(e + \frac{1}{e}\right) - 2\sum_{k=0}^{n} \frac{1}{(2k)!}\right| < \frac{2}{(2n)!}.$$

■ **Solution** Applying Proposition 4.19 with $f(x) = e^x$, $a = 0$ and order $2n$, we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n}}{(2n)!} + \frac{1}{(2n)!}\int_0^x (x-t)^{2n} e^t \, dt,$$
$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n}}{(2n)!} + \frac{1}{(2n)!}\int_0^x (x-t)^{2n}(-e^{-t}) \, dt.$$

Putting $x = 1$ and adding the above results, we get:

$$e + \frac{1}{e} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!}\right) + \frac{1}{(2n)!}\int_0^1 (1-t)^{2n}(e^t - e^{-t}) \, dt.$$

Next we proceed to estimate:

$$\begin{aligned}
&\left|\left(e + \frac{1}{e}\right) - 2\sum_{k=0}^{n} \frac{1}{(2k)!}\right| \\
&= \frac{1}{(2n)!}\left|\int_0^1 (1-t)^{2n}(e^t - e^{-t}) \, dt\right| \\
&\le \frac{1}{(2n)!}\int_0^1 \left|(1-t)^{2n}\right| \left|e^t - e^{-t}\right| \, dt \\
&\le \frac{1}{(2n)!}\int_0^1 (e^t - e^{-t}) \, dt && \text{as } \left|(1-t)^{2n}\right| \le 1 \text{ when } t \in [0,1] \\
&= \frac{1}{(2n)!}[e^t + e^{-t}]_0^1 \\
&= \frac{1}{(2n)!}(e^1 + e^{-1} - 2).
\end{aligned}$$

It is well-known that $2 < e < 3$, so $e + \frac{1}{e} - 2 < 3 + \frac{1}{2} - 2 < 2$. It proves:

$$\left|\left(e + \frac{1}{e}\right) - 2\sum_{k=0}^{n} \frac{1}{(2k)!}\right| < \frac{2}{(2n)!}$$

as desired.

■ **Exercise 4.46** Assume all conditions given in Proposition 4.19. Use the proposition to prove that if $|f(x)| \le M$ on any interval I containing a, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for any $x \in I$.

■ **Exercise 4.47 — Source: HKAL 1993 Paper II Q12 (modified).** (a) Show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2} \, dt$$

for all $x \in \mathbb{R}$ and $n = 1, 2, 3, \cdots$.

(b) Using (a), or otherwise, show that

$$\left|\tan^{-1} x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1}\right)\right| \le \frac{x^{2n+1}}{2n+1}$$

for all $x \geq 0$ and $n = 1, 2, 3, \cdots$. Hence find $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.

(c) Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$. Deduce that

$$\left| \frac{\pi}{4} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}\left(\frac{1}{2^{2k-1}} + \frac{1}{3^{2k-1}}\right) \right| \leq \frac{1}{n \cdot 2^{2n+1}}$$

for $n = 1, 2, 3, \cdots$.

4.5.4 Young's and Hölder's inequalities

In this subsection we use integration by parts and by substitutions to derive the Young's inequality, which is a generalization of the trivial result $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \geq 0$. We first prove the following integral inequality:

Proposition 4.20 Given $c > 0$, $f : [0, c] \to \mathbb{R}$ is a strictly increasing differentiable function on $[0, c]$, and $f(0) = 0$. Then, for all $a \in [0, c]$ and $b \in [0, f(c)]$,

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$$

with equality holds if and only if $b = f(a)$. The geometric meaning of the inequality can be found in Figure 4.5.

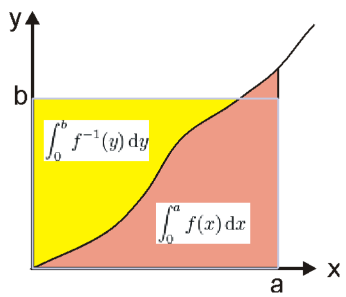


Figure 4.5: Graphical meaning of Proposition 4.20

Proof. By intermediate value theorem, we can take $b = f(\varphi)$ for some $\varphi \in [0, c]$. Consider the right integral and let $z = f^{-1}(y)$, then when $y = 0$, $z = 0$; and when $y = b = f(\varphi)$, $z = \varphi$. Also, we have

$$f(z) = y \implies f'(z)\,dz = dy.$$

Using integration by substitution and then by parts, we get:

$$\begin{aligned}\int_0^b f^{-1}(y)\,dy &= \int_{z=0}^{z=\varphi} z f'(z)\,dz = \int_{x=0}^{x=\varphi} x f'(x)\,dx \\ &= [x f(x)]_0^{\varphi} - \int_0^{\varphi} f(x)\,dx = \varphi b - \int_0^{\varphi} f(x)\,dx.\end{aligned}$$

We first assume $\varphi \leq a$, then as f is increasing, we have $f(x) \geq f(\varphi) = b$ for any $x \in [\varphi, a]$, and so

$$\begin{aligned}\int_0^a f(x)\,dx &= \int_0^{\varphi} f(x)\,dx + \int_{\varphi}^a f(x)\,dx \\ &\geq \int_0^{\varphi} f(x)\,dx + \int_{\varphi}^a b\,dx = \int_0^{\varphi} f(x)\,dx + b(a - \varphi).\end{aligned}$$

Adding both results, we get:

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq \varphi b + (a-\varphi)b = ab.$$

Equality holds if and only if $f(x) = b$ for all $x \in [\varphi, a]$. However, f is strictly increasing, so it would happen only when $a = \varphi$ (equivalently, $f(a) = f(\varphi) = b$).

We leave it as an exercise for readers to prove the case $\varphi > a$. ■

■ **Exercise 4.48** Complete the proof of the case $\varphi > a$. Hint: draw a diagram to get some geometric idea.

■ **Exercise 4.49 — Source: MATH1024 Spring 2018 Midterm.** Consider a bijective function $f : [a,b] \to [f(a), f(b)]$ where $b > a > 0$ and $f(b) > f(a) > 0$, and given that f is differentiable on $[a,b]$ and $f'(x) > 0$ on (a,b).

(a) By sketching a diagram, guess the value of:

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

in terms of $a, b, f(a), f(b)$.

(b) Prove your claim in (a) using integration by substitution.

(c) Let $g(x) = 5\sqrt{x} - 6$. Show that the definite integral:

$$\int_4^9 g(g(g(g(g(x)))))\,dx$$

is a rational number.

Corollary 4.21 — Young's Inequality. For any $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds if and only if $b = a^{1/p}$.

Proof. Note that the result is trivial if one of a, b is zero. We now assume $a, b > 0$. We just apply Proposition 4.20 on the function $f(x) = x^p$, whose derivative is $f'(x) = px^{p-1} > 0$ on $(0, \infty)$. The inverse function is given by $f^{-1}(x) = x^{1/p} = x^{1-1/q}$. It can be shown easily that:

$$\int_0^a f(x)\,dx = \frac{a^p}{p} \quad \text{and} \quad \int_0^b f^{-1}(x)\,dx = \frac{b^q}{q}.$$

■

Young's inequality can be used to prove another (even more important) inequality, the Hölder's inequality, which plays a crucial role in functional analysis. For simplicity, we first denote for each $p \geq 1$ the l_p-norm of a finite sequence $\{x_n\}_{n=1}^N$ and the L_p-norm of a continuous function f on $[a,b]$ by:

$$\|\{x_n\}\|_p := \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}}$$

$$\|f\|_p := \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}$$

It is clear that for any $c \in \mathbb{R}$, $\|\{cx_n\}\|_p = |c| \|\{x_n\}\|_p$ and $\|cf\|_p = |c| \|f\|_p$, and so $\frac{\{x_n\}}{\|\{x_n\}\|_p}$ and $\frac{f}{\|f\|_p}$ have unit l_p- and L_p-norms. The Hölder's inequality is the following:

Proposition 4.22 — Hölder's Inequality. Given $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then we have:

$$\|\{x_n y_n\}\|_1 \le \|\{x_n\}\|_p \|\{y_n\}\|_q$$
$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

for any finite sequences $\{x_n\}$ and $\{y_n\}$ and any continuous functions f and g on $[a, b]$.

Proof. We prove the Hölder's inequality for functions and leave the sequence's version as an exercise for readers. Using the Young's inequality with $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$, we have:

$$\frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q} \le \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q \quad \forall x \in [a, b].$$

Therefore, by integrating both sides over $[a, b]$, we get:

$$\int_a^b \frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q}\, dx \le \int_a^b \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p dx + \int_a^b \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q dx.$$

Note that the norms are all constants, so we have:

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \le \frac{1}{p\|f\|_p^p} \int_a^b |f(x)|^p\, dx + \frac{1}{q\|g\|_q} \int_a^b |g(x)|^q\, dx.$$

Note that

$$\|f\|_p^p = \int_a^b |f(x)|^p\, dx$$

and similarly for g, so we get

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1,$$

and so our desired result holds.

Note that we assumed f and g are not identically 0 in the above proof. In one of them is identically zero, the result is trivial. ■

■ **Exercise 4.50** Prove the sequence version of the Hölder's inequality.

■ **Exercise 4.51** When does the equality hold for the Hölder's inequality?

Clearly, the Hölder's inequality is a generalization of the well-known Cauchy-Schwarz's inequality. The latter is a special case $p = q = 2$.

■ **Exercise 4.52 — Source: HKAL 2002 Paper I Q8.** Answer the following questions:

(a) Proof of the Cauchy-Schwarz's inequality (omitted here).

(b) (i) Prove that

$$\left(\frac{\sum_{i=1}^n x_i}{n}\right)^2 \le \frac{\sum_{i=1}^n x_i^2}{n},$$

where $x_1, x_2, \cdots, x_n$ are real.

(ii) Prove that

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \lambda_i x_i^2\right),$$

where $x_1, x_2, \cdots, x_n$ are real numbers and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are positive numbers. When does the equality hold?

(iii) Prove that

$$\left(\frac{y_1}{t} + \frac{y_2}{t^2} + \cdots + \frac{y_n}{t^n}\right)^2 < \frac{y_1^2}{t} + \frac{y_2^2}{t^2} + \cdots + \frac{y_n^2}{t^n},$$

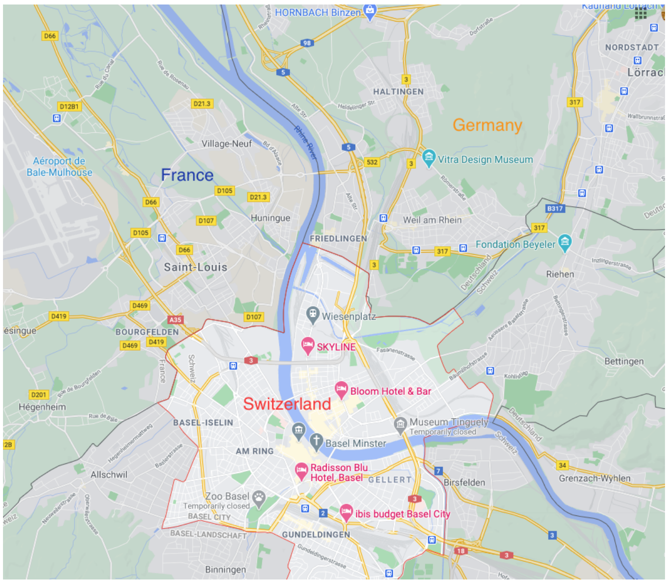
where $y_1, \cdots, y_n$ are real numbers, not all zero, and $t \geq 2$.

4.5.5 Basel Problem

The Basel Problem is about finding the exact value of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

It was first posed by Pietro Mengoli in 1644, and was first solved by Euler in 1734 using infinite products. He found that the exact value of this infinite sum is $\frac{\pi^2}{6}$. The name of the problem, Basel, is the name of a city in Switzerland near the border with France and Germany. The city is the hometown of Euler and the Bernoulli's family.



The original proof of Euler used infinite products which will not be discussed in the course, but there are some other proof using different techniques. Some used more advanced tools such as Fourier series and complex analysis. Below are two of the proofs that can be understood with some basic knowledge about integration by parts. They are restructured as two exercises below:

Exercise 4.53 — Source: MATH1024 Spring 2018 Final Exam. For each integer $n \geq 0$, we define

$$A_n := \int_0^{\pi/2} \cos^{2n} x \, dx \qquad B_n := \int_0^{\pi/2} x^2 \cos^{2n} x \, dx.$$

(a) Show that for any integer $n \geq 1$, we have $2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) = \frac{1}{n^2}$.

(b) Show that there exists a constant $C > 0$, independent of n, such that

$$B_n \leq \frac{C}{n+1} A_n$$

for any integer $n \geq 1$. [Hint: Compare $\sin x$ with a linear function on $0 \leq x \leq \frac{\pi}{2}$.]

(c) Using the above results, show that $\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

■ **Exercise 4.54** Define the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ and the function $g(x)$:

$$f_n(x) := \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx$$
$$g(x) := \begin{cases} \frac{x/2}{\sin(x/2)} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Consider the integral:

$$E_n := \int_0^{\pi} x f_n(x)\, dx.$$

(a) Show that

$$E_n = \frac{\pi^2}{4} + \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2},$$

and so $E_{2n-1} = \frac{\pi^2}{4} - 2\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$ for any $n \in \mathbb{N}$.

(b) Show that g is a C^1 function, and that

$$E_{2n-1} = \frac{1}{4n-1}\left(2 + 2\int_0^{\pi} g'(x) \cos \frac{(4n-1)x}{2}\, dx\right)$$

(c) Prove that $E_{2n-1} \to 0$ as $n \to \infty$, and show $\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

[Remark: We may need the fact that rearrangement of a convergent series of positive numbers preserves its value. We will prove it later.]

4.5.6 Irrationality of π

In this subsection we present two proofs of irrationality of π. They are again restructured as two exercises.

■ **Exercise 4.55 — Source: Exam at Cambridge University 1945, written by Mary Cartwright.** Consider the sequence of functions of x:

$$I_n(x) := \int_{-1}^{1} (1 - z^2)^n \cos(xz)\, dz,$$

where $n \in \mathbb{N} \cup \{0\}$.

(a) Show that for any $n \geq 2$ and $x \in \mathbb{R}$, we have:

$$x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x).$$

(b) Define $J_n := x^{2n+1} I_n(x)$. Prove that for any $n \in \mathbb{N}$,

$$J_n(x) = n!(P_n(x)\sin(x) + Q_n(x)\cos(x))$$

where P_n and Q_n are polynomials of integer coefficients and $\deg P_n, \deg Q_n \leq n$.

(c) Now assume $\pi \in \mathbb{Q}$ and write $\pi = \frac{2a}{b}$ where $a, b \in \mathbb{N}$. Note that we do not assume $\frac{2a}{b}$ is in the simplest form so we can assume that numerator is even. Verify the following:

(i) For any $n \in \mathbb{N}$, we have

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) = P_n(\pi/2) b^{2n+1}.$$

(ii) Deduce a contradiction by showing that LHS $\to 0$ as $n \to \infty$ whereas RHS is always a positive integer.

■ **Exercise 4.56 — Source: Exercise in a Bourbaki's book**[a]. For each $n \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$, we define

$$A_n(b) := b^n \int_0^\pi \frac{x^n(\pi - x)^n}{n!} \sin x \, dx.$$

(a) Prove that for each $b \in \mathbb{N}$, we have $0 < A_n(b) < 1$ for sufficiently large n.

(b) Now suppose π is rational and $\pi = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. Consider the polynomial $f(x) := \dfrac{x^n(a - bx)^n}{n!}$, prove that:

$$A_n(b) = [-f(x)\cos x]_0^\pi - [-f'(x)\sin x]_0^\pi + \cdots \pm \Big[f^{(2n)}(x)\cos x\Big]_0^\pi \pm \int_0^\pi f^{(2n+1)}(x)\cos x \, dx.$$

(c) Hence, by showing $A_n(b)$ is an integer, deduce a contradiction.

[a] Bourbaki is a **group** of prominent mathematicians, including Cartan and Weil, who co-authored a huge collection of books and treaties in various topics of pure mathematics. There is one related joke: "When did Bourbaki stop writing books? Answer: After they realized that Serge Lang is a single person."