4.4 Integration by Substitutions

In this and the next sections we discuss some common techniques of doing integrations, including method of substitutions and integration by parts. Let's start with the method of substitutions:

Proposition 4.13 Suppose $u = g(x) : [a, b] \to I$ is C^1 function on $x \in [a, b]$, and $f(u)$ is continuous on $u \in I$ with an anti-derivative F, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. By the chain rule, we have

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

Therefore, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $x \in [a, b]$, and we have

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

Moreover,

$$\int_{u=g(a)}^{u=g(b)} f(y)\,dy = F(g(b)) - F(g(a)).$$

Combining the results, we have

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

■

An easy to remember this rule is to regard $g'(x)\,dx$ as du (by the virtue of $du = \frac{du}{dx}\,dx$, and $f(g(x))$ as $f(u)$. Also $x = a, b$ corresponds to $u = g(a), g(b)$ respectively.

■ **Exercise 4.31** Prove the indefinite integral version of the substitution rule:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Here we implicitly assume that x and u lie on some intervals on which the conditions in Proposition 4.13 hold.

■ **Example 4.9** Consider

$$\int_0^2 x(2x^2+3)^2\,dx.$$

We let $f(u) = u^2$ and $g(x) := 2x^2 + 3$, then $g'(x) = 4x$, so we write

$$\int_0^2 x(2x^2+3)^2\,dx = \frac{1}{4}\int_0^2 \underbrace{(2x^2+3)^2}_{=f(g(x))} \cdot \underbrace{4x}_{=g'(x)}\,dx = \frac{1}{4}\int_{g(0)}^{g(2)} \underbrace{u^2}_{=f(u)}\,du = \frac{1}{4}\left[\frac{u^3}{3}\right]_3^{11} = \frac{11^3-3^3}{12}.$$

Very often, we would write the above solution without defining so many functions f and g but simply let $u = 2x^2 + 3$. Instead of "creating" a term $g'(x)\,dx$, we compute

$$du = \frac{du}{dx}\,dx = 4x\,dx \implies x\,dx = \frac{1}{4}\,du,$$

and when $x = 0$, $u = 3$; whereas when $x = 2$, $u = 11$. These combine to give:

$$\int_0^2 x(2x^2+3)^2\,dx = \int_0^2 \underbrace{(2x^2+3)^2}_{=u^2} \cdot \underbrace{x\,dx}_{=\frac{1}{4}\,du} = \int_3^{11} \frac{1}{4}u^2\,du = \frac{11^3 - 3^3}{12}.$$

For a simple integral like this example, we may even save the use of the letter u and just write:

$$x\,dx = \frac{1}{2}d(x^2) = \frac{1}{4}d(2x^2+3),$$

and so we have

$$\int_0^2 x(2x^2+3)^2\,dx = \frac{1}{4}\int_{x=0}^{x=2} (2x^2+3)^2\,d(2x^2+3).$$

Then we just regard $2x^2 + 3$ as the integration variable, and simply integrate the square function:

$$\frac{1}{4}\int_{x=0}^{x=2} (2x^2+3)^2\,d(2x^2+3) = \frac{1}{4}\left[\frac{(2x^2+3)^3}{3}\right]_{x=0}^{x=2} = \frac{11^3 - 3^3}{4}.$$

Comparing the three ways of maneuvering the integration by substitution, the first one is seldom used – it is only good for giving the precise statement of Proposition 4.13. The second and third ones are more common, and the third one is often used for simple substitutions.

■ **Exercise 4.32** Compute the following integrals:

1. $\displaystyle\int_a^b x\cos(x^2+1)\,dx$
2. $\displaystyle\int_a^b x^3 e^{x^4}\,dx$
3. $\displaystyle\int_a^b \frac{x}{1+x^2}\,dx$

4.4.1 Trigonometric functions

Many integration formulae of some trigonometric functions are derived using substitutions. Below we will state the indefinite integral version. They can be applied to definite integrals as long as the integrand is continuous on the integration interval.

Proposition 4.14

$$\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$$

$$\int \cot x\,dx = \log|\sin x| + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\log|\csc x + \cot x| + C$$

Proof. We prove only the formulae for $\tan x$ and $\sec x$, and leave the other two as an exercise.

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{1}{\cos x}\,d(\cos x) && \text{(implicitly letting } u = \cos x\text{)} \\ &= -\log|\cos x| + C \\ &= \log\left|\frac{1}{\cos x}\right| + C = \log|\sec x| + C.\end{aligned}$$

The formula for $\sec x$ involves a somewhat clever observation:

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx \\ &= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{1}{\sec x + \tan x}\,d(\tan x + \sec x) \\ &= \log|\sec x + \tan x| + C.\end{aligned}$$

■

■ **Exercise 4.33** Prove that

$$\int \cot x\,dx = \log|\sin x| + C$$
$$\int \csc x\,dx = -\log|\csc x + \cot x| + C$$

To apply the above integral formulae on definite integrals, we need to make sure the function is continuous on interval of integration.

$$\int_0^{\pi/4} \tan x\,dx = [\log|\sec x|]_0^{\pi/4} = \log\sqrt{2} \qquad \text{(RIGHT)}$$
$$\int_0^{\pi} \tan x\,dx = [\log|\sec x|]_0^{\pi} = 0 \qquad \text{(WRONG!)}$$

4.4.2 Trigonometric substitutions

Below are more examples of integration formulae:

Proposition 4.15

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

where $a > 0$.

Proof. For the $\frac{1}{\sqrt{a^2-x^2}}$ integral, we write $x = a\sin u$, which means we let $u = \sin^{-1}\frac{x}{a}$. The range of u is then $(-\pi/2, \pi/2)$. Then, we have

$$dx = d(a\sin u) = a\cos u\,du.$$

The integrand becomes

$$\frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{a^2 - a^2\sin^2 u}} = \frac{1}{\sqrt{a^2\cos^2 u}} = \frac{1}{|a\cos u|} = \frac{1}{a\cos u}$$

as $a > 0$ and $\cos u > 0$ when $u \in (-\pi/2, \pi/2)$. These show

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{a\cos x}\, a\cos u\, du = \int 1\, du = u + C = \sin^{-1}\frac{x}{a} + C.$$

For the $\frac{1}{a^2+x^2}$ integral, we write $x = a\tan u$, which means $u = \tan^{-1}\frac{x}{a}$. Then, we have

$$dx = a\sec^2 u\, du \text{ and } \frac{1}{a^2+x^2} = \frac{1}{a^2(1+\tan^2 u)} = \frac{1}{a^2\sec^2 u}.$$

Combining both, we get:

$$\int \frac{1}{a^2+x^2}\, dx = \int \frac{1}{a^2\sec^2 u}\, a\sec^2 u\, du = \int \frac{1}{a}\, du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

■

ⓘ The choice of the trigonometric functions above is motivated by the formulae $\sin^2 x + \cos^2 = 1$ and $1 + \tan^2 x = \sec^2 x$.

■ **Exercise 4.34** Prove that integration formula below:

$$\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log\left|x + \sqrt{x^2+a^2}\right| + C$$

$$\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

■ **Exercise 4.35** Show that for any $r > 0$, we have

$$\int_0^r \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{4}.$$

This shows the area of the circle with radius r is πr^2.

4.4.3 More uses of integration by substitutions

One can also use the integration by substitution to prove some general results about definite integrals.

Proposition 4.16 Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic continuous function of period T, i.e. $f(x+T) = f(x)$ for any $x \in \mathbb{R}$. Show that

$$\int_a^b f(x)\, dx = \int_{a+T}^{b+T} f(x)\, dx.$$

Proof. Let $u = x + T$, then $dx = du$. When $x = a$, $u = a + T$; and when $x = b$, $u = b + T$. Therefore, we have

$$\int_a^b f(x)\, dx = \int_a^b f((x+T) - T)\, dx = \int_{u=a+T}^{u=b+T} f(u-T)\, du = \int_{u=a+T}^{u=b+T} f(u)\, du.$$

Here we have used the fact that $f(u - T) = f(u - T + T) = f(u)$. Note that the u in the last integral is dummy, so we can change it back to x:

$$\int_{u=a+T}^{u=b+T} f(u)\, du = \int_{x=a+T}^{x=b+T} f(x)\, dx.$$

This completes our proof. ■

Proposition 4.17 For any continuous odd function f, i.e. $f(-x) = -f(x)$ for any $x \in \mathbb{R}$, we have:

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{for any } a > \mathbb{R}.$$

For any continuous even function g, i.e. $g(-x) = g(x)$ for any $x \in \mathbb{R}$, we have:

$$\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx \quad \text{for any } a > 0.$$

Proof. For the odd function result, we need to show

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx.$$

We let $u = -x$, then when $x = 0$, $u = 0$; and when $x = -a$, $u = a$. Therefore,

$$\int_{x=-a}^{x=0} f(x)\,dx = \int_{u=a}^{u=0} f(-u)(-\,du) = \int_{a}^{0} -f(u)(-\,du) = -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx.$$

For the even function result, we need to show

$$\int_{-a}^{0} g(x)\,dx = \int_{0}^{a} g(x)\,dx.$$

The proof is very similar: let $u = -x$, then

$$\int_{x=-a}^{x=0} g(x)\,dx = \int_{u=a}^{u=0} g(-u)(-\,du) = \int_{a}^{0} g(u)(-\,du) = \int_{0}^{a} g(u)\,du = \int_{0}^{a} g(x)\,dx.$$

■

■ **Example 4.10 — Source: HKAL 2002 Paper II9, excerpt.** Let $f : \mathbb{R} \to [0, \infty)$ be a periodic function with period T.

(a) Prove that

$$\int_{a+kT}^{b+kT} e^{-x} f(x)\,dx = e^{-kT}\int_{a}^{b} e^{-x} f(x)\,dx$$

for any $k \in \mathbb{N}$.

(b) Let $I_n = \int_{0}^{nT} e^{-x} f(x)\,dx$. Prove that

$$I_n = \frac{1 - e^{-nT}}{1 - e^{-T}} I_1$$

for any $n \in \mathbb{N}$.

(c) If $l > 0$ and n is the positive integer such that $nT \le l < (n+1)T$, prove that

$$\frac{1 - e^{-nT}}{1 - e^{-T}} I_1 \le \int_{0}^{l} e^{-x} f(x)\,dx \le \frac{1 - e^{-(n+1)T}}{1 - e^{-T}} I_1.$$

■ **Solution** (a) Consider the integral $\int_{a}^{b} e^{-x} f(x)\,dx$. Let $u = x + kT$, then $du = dx$; and when

$x = a$, $u = a + kT$; when $x = b$, $u = b + kT$. Therefore,

$$\begin{aligned}
\int_a^b e^{-x} f(x)\, dx &= \int_{u=a+kT}^{u=b+kT} e^{-(u-kT)} f(u-kT)\, du \\
&= \int_{a+kT}^{b+kT} e^{kT} e^{-u} f(u)\, du \qquad \text{(since } f(u-kT) = f(u)\text{)} \\
&= \underbrace{e^{kT}}_{\text{constant}} \underbrace{\int_{a+kT}^{b+kT} e^{-x} f(x)\, dx}_{\text{change dummy vars}}
\end{aligned}$$

By rearrangement, we get:

$$\int_{a+kT}^{b+kT} e^{-x} f(x)\, dx = e^{-kT} \int_a^b e^{-x} f(x)\, dx.$$

(b) Note that

$$\begin{aligned}
I_n &= \int_0^T e^{-x} f(x)\, dx + \int_T^{2T} e^{-x} f(x)\, dx + \cdots + \int_{(n-1)T}^{nT} e^{-x} f(x)\, dx \\
&= I_1 + \int_{0+T}^{T+T} e^{-x} f(x)\, dx + \cdots + \int_{0+(n-1)T}^{T+(n-1)T} e^{-x} f(x)\, dx \\
&= I_1 + e^{-T} \int_0^T e^{-x} f(x)\, dx + e^{-2T} \int_0^T e^{-x} f(x)\, dx + \cdots + e^{-(n-1)T} \int_0^T e^{-x} f(x)\, dx \\
&= I_1 + e^{-T} I_1 + e^{-2T} I_1 + \cdots + e^{-(n-1)T} I_1 \\
&= \frac{I_1(1-(e^{-T})^n)}{1-e^{-T}}.
\end{aligned}$$

The last step used the geometric series formula with common ratio e^{-T}. This proves the result in (b).
(c) Note that $e^{-x} f(x) \geq 0$ as given. Therefore,

$$\underbrace{\int_0^{nT} e^{-x} f(x)\, dx}_{I_n} \leq \int_0^l e^{-x} f(x)\, dx \leq \underbrace{\int_0^{(n+1)T} e^{-x} f(x)\, dx}_{I_{n+1}}$$

From (b), we conclude that

$$\frac{1-e^{-nT}}{1-e^{-T}} I_1 \leq \int_0^l e^{-x} f(x)\, dx \leq \frac{1-e^{-(n+1)T}}{1-e^{-T}} I_1$$

as desired.

■ **Exercise 4.36 — Source: HKAL 2012 Paper II Q8.** Answer the following questions:

(a) (i) Prove that $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin x}\, dx = 1$.

(ii) Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\sin x}\, dx$.

(b) Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function such that $f(\pi - x) = f(x)$ for all $x \in [0, \pi]$. Using integration by substitution, prove that

$$\int_0^{\pi} f(x)\, dx = 2 \int_0^{\pi/2} f(x)\, dx.$$

(c) Let $g : [0, \pi] \to \mathbb{R}$ be a continuous function such that $g(\pi - x) = -g(x)$ for all $x \in [0, \pi]$. Using the substitution $u = \pi - x$, prove that

$$\int_0^\pi g(x) \log(1 + e^{\cos x})\, dx = \frac{1}{2} \int_0^\pi g(x) \cos x \, dx.$$

(d) Evaluate $\displaystyle\int_0^\pi \frac{\cos x \cdot \log(1 + e^{\cos x})}{(1 + \sin x)^2}\, dx$.

■ **Exercise 4.37** Consider the integral:

$$I_a := \int_0^\pi \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}\, d\theta$$

where $a \in (0, \infty)\backslash\{1\}$. This integral appears in the calculation of electric flux across a unit sphere with a point charge either inside $(a < 1)$ or outside $(a > 1)$ the sphere. One elegant way of computing I_a is to use complex analysis. This exercise is about a less elegant, but more elementary, approach of evaluating I_a.

(a) Show that for any $a \in (0, \infty)\backslash\{1\}$ and $\theta \in (0, \pi)$.

$$\frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2} = \frac{1}{2} + \frac{1 - a}{1 + a} \cdot \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{\left(\frac{1-a}{1+a}\right)^2 + \tan^2\frac{\theta}{2}}$$

(b) Note that $\sec^2\frac{\theta}{2}$ and $\tan^2\frac{\theta}{2}$ are not both well-defined at $\theta = 0, \pi$, but $\frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}$ is defined and is continuous on the whole interval $[0, \pi]$. Let $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in (\frac{\pi}{2}, \pi)$. Using (a), compute

$$\int_\alpha^\beta \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}\, d\theta.$$

(c) Hence, show that

$$I_a = \begin{cases} \pi & \text{if } a < 1 \\ 0 & \text{if } a > 1 \end{cases}$$