4.3 Fundamental Theorem of Calculus

4.3.1 Newton-Leibniz's formula

In previous sections we have established the rigorous definition of Riemann integrals. In particular, we proved that any continuous function on $[a, b]$ must be Riemann integrable. However, it is rather impractical to compute $\displaystyle\int_a^b f(x)\,dx$ via taking a sequence of partitions $\{P_n\}$, as we have seen that even the computation of $\displaystyle\int_0^1 x^p\,dx$ where $p \in \mathbb{N}$ could involve some summation formulae.

The Fundamental Theorem of Calculus links the Riemann integral of a continuous function with its anti-derivatives, and provides us a very effective way of computing the value of the integral.

Theorem 4.11 — Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$, then we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{for any } x \in [a, b]. \tag{4.2}$$

Furthermore, if F is a differentiable function such that $F'(x) = f(x)$ for any $x \in [a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{4.3}$$

(4.3) is known as the **Newton-Leibniz's Formula**. The function F is called an **anti-derivative**, or a **primitive function**, of f.

Note that $\int_a^x f(t)\,dt$ is a function of x, not of t. We use t inside the integral $\int_a^x f(t)\,dt$ because x has appeared as the upper bound of the integral $\int_a^x$. You can use any other variable too (except x). We usually call t as the *dummy variable*.

Proof. To prove (4.2), we consider the definition of derivatives

$$\begin{aligned}\frac{d}{dx}\int_a^x f(t)\,dt &= \lim_{h\to 0}\frac{\displaystyle\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt.\end{aligned}$$

The last step follows from (1) of Proposition 4.10.

By the result of Example 4.7, there exists c between x and $x + h$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{(x+h)-x}\int_x^{x+h} f(t)\,dt = f(c).$$

Note that this c depends on both x and h.

Letting $h \to 0$ (keeping x fixed), by $c \in [x, x+h]$ or $[x+h, x]$, we have $c \to x$ and so by continuity of f we get

$$\lim_{h\to 0} f(c) = f(x).$$

This proves (4.2).

For (4.3), we consider

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt - F(x)\right) = f(x) - f(x) = 0 \quad \text{for any } x \in [a, b]$$

according to (4.2) and the given condition about F.

The only functions with derivatives are constant function (a consequence of the mean value theorem). Therefore, there exists $C \in \mathbb{R}$ such that

$$\int_a^x f(t)\,dt - F(x) = C \quad \text{for any } x \in [a,b].$$

In particular, putting $x = a$ we get:

$$\underbrace{\int_a^a f(t)\,dt}_{=0} - F(a) = C \implies C = -F(a).$$

Therefore, we have

$$\int_a^x f(t)\,dt = F(x) + C = F(x) - F(a) \quad \text{for any } x \in [a,b],$$

and in particular by putting $x = b$, we get (4.3). ■

We often denote $F(b) - F(a)$ by $[F(x)]_a^b$, $F(x)|_a^b$.

Using (4.3), we can compute the integrals appeared in the previous section very easily – simply find an anti-derivative.

$$\frac{d}{dx}\frac{x^{p+1}}{p+1} = x^p \text{ where } p \geq 0 \implies \int_0^1 x^p\,dx = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{1^{p+1}}{p+1} - \frac{0^{p+1}}{p+1} = \frac{1}{p+1}$$

$$\frac{d}{dx}(-\cos x) = \sin x \implies \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = 2$$

$$\frac{d}{dx}e^x = e^x \implies \int_a^b e^x\,dx = [e^x]_a^b = e^b - e^a$$

Continuity is crucial when applying the Newton-Leibniz's formula. The following absurd result would come up if one applies (4.3) blindly on a discontinuous function:

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^1 = -2 \qquad \text{(WRONG!)}$$

Clearly $\frac{1}{x^2} > 0$, so it is absurd for its Riemann integral being negative! The pitfall is that $\frac{1}{x^2}$ is not continuous at 0 which lies in the interval $[-1,1]$. We cannot apply (4.3) directly!

However, it is perfectly fine to use (4.3) on

$$\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2},$$

as $\frac{1}{x^2}$ is continuous on $[1,2]$. We will discuss $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ later as the function is unbounded on $[0,1]$. It is an improper integral.

■ **Exercise 4.23** Find the value of each integral below using Newton-Leibniz's formula:

1. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$
2. $\displaystyle\int_0^\pi x\cos(x^2)\,dx$
3. $\displaystyle\int_a^b \sin(Ax+B)\,dx$ where $A \neq 0$ and B are constants.

4.3.2 More uses of the Fundamental Theorem of Calculus

Let's discuss more about the use of (4.2). First note that we stated (4.2) as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

the lower bound a of the integral can be replaced by any other constant c as

$$\int_c^x f(t)\,dt = \int_a^x f(t)\,dt - \int_a^c f(t)\,dt$$

and $\int_a^c f(t)\,dt$ is a constant.

However, one should note that the upper bound of the integral must be $\int^x$ otherwise one should consider using the chain rule. Another issue is that (4.2) requires the integrand $f(t)$ to be independent of the differentiate variable x. Let's see some examples:

■ **Example 4.8** Find the derivative with respect to x of each function below. Assume that f is continuous on $\mathbb{R}$.

1. $F(x) = \int_0^{x^2} f(t)\,dt$
2. $G(x) = \int_a^x x f(t)\,dt$
3. $H(x) = \int_x^{x^2} f(t)\,dt$

■ **Solution** The upper bound of the integral for $F(x)$ is $\int^{x^2}$, we should use the chain rule:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_0^{x^2} f(t)\,dt \\ &= \frac{d}{d(x^2)}\int_0^{x^2} f(t)\,dt \cdot \frac{d}{dx}x^2 \\ &= f(x^2)\cdot 2x = 2xf(x^2). \end{aligned}$$

For $G(x)$, the integrand $xf(t)$ depends on x, so one must take it out from the integral first before applying (4.2):

$$\int_a^x xf(t)\,dt = x\int_a^x f(t)\,dt.$$

The above holds because x is independent of the integration variable t. Then,

$$\begin{aligned} G'(x) &= \frac{d}{dx}\left(x\int_a^x f(t)\,dt\right) \\ &= \frac{dx}{dx}\int_a^x f(t)\,dt + x\frac{d}{dx}\int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + xf(x). \end{aligned}$$

We cannot proceed further because f is not explicitly given.

For $H(x)$, note that the lower bound is also a function of x, so we first rewrite the integral as:

$$H(x) = \int_x^{x^2} f(t)\,dt = \underbrace{\int_0^{x^2} f(t)\,dt}_{G(x)} - \int_0^x f(t)\,dt.$$

You can replace 0 by any other number provided that f is continuous on the interval of integration. Then we have:

$$H'(x) = G'(x) - f(x) = 2xf(x^2) - f(x).$$

■ **Exercise 4.24** Derive a formula for:

$$\frac{d}{dx}\int_{\beta(x)}^{\alpha(x)} f(t)\,dt$$

where f is continuous on $\mathbb{R}$, and α, β are differentiable on $\mathbb{R}$.

■ **Exercise 4.25** Let $f : \mathbb{R} \to (0, \infty)$ be continuous function, and consider

$$g(x) := \frac{\left(\int_0^x tf(t)\,dt\right)^2}{\int_0^x f(t)\,dt}.$$

Prove that g is strictly increasing on $(0, \infty)$.

■ **Exercise 4.26 — Source: HKAL 1994.** Let $f(x) = \int_1^x \sin(\cos t)\,dt$.

(a) Show that f is injective on $[0, \pi/2)$.

(b) Find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=0}$

■ **Exercise 4.27 — Source: HKAL 1997.** Evaluate

$$\lim_{x\to 0+}\left(\frac{1}{x^3}\int_0^x e^{t^2}\,dt - \frac{1}{x^2}\right).$$

■ **Exercise 4.28** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f satisfies the differential equation

$$f'(x) = \sin\left(1 + f(x)^2\right) \text{ and } f(0) = a$$

if and only if f satisfies the integral equation

$$f(x) = a + \int_0^x \sin\left(1 + f(t)^2\right)\,dt.$$

Let's discuss more use of (4.2):

Proposition 4.12 Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function. Suppose

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) \equiv 0$ on $[a, b]$.

Proof. It is quite an expected result since $\int_a^b f(x)\,dx$ is the area under the graph $y = f(x)$ for a non-negative function f. If the area is zero, the only possibility is the function is 0. As we also assume f is continuous, we rule out those function which is 0 except a finite number of point too.

To prove it rigorously, we consider the function

$$F(t) := \int_a^t f(x)\,dx.$$

By (4.2), we have $F'(t) = f(t) \geq 0$. Hence F is increasing on $[a, b]$. However, we also note that

$$F(a) = \int_a^a f(x)\,dx = 0 \quad \text{and} \quad F(b) = \int_a^b f(x)\,dx = 0 \quad \text{(given)}.$$

Therefore, $F(t)$ is identically zero on $[a, b]$ since:

$$0 = F(a) \leq F(t) \leq F(b) = 0 \; \forall t \in [a, b].$$

This prove $f(t) = F'(t) = 0$ on $[a, b]$. ■

■ **Exercise 4.29 — Source: HKAL 1998.** Answer the following questions:

(a) [This part just asked for the proof of Proposition 4.12, hence omitted here.]

(b) Let g be a continuous function on $[a, b]$. Suppose

$$\int_a^b g(x)u(x)\,dx = 0$$

for any continuous function u on $[a, b]$, show that $g(x) = 0$ for all $x \in [a, b]$.

(c) Let h be a continuous function on $[a, b]$. Define

$$A = \frac{1}{b-a}\int_a^b h(t)\,dt.$$

(i) If $v(x) = h(x) - A$ for all $x \in [a, b]$, show that $\int_a^b v(x)\,dx = 0$.

(ii) If $\int_a^b h(x)w(x)\,dx = 0$ for any continuous function w on $[a, b]$ satisfying $\int_a^b w(x)\,dx = 0$, show that $h(x) = A$ for all $x \in [a, b]$.

4.3.3 Indefinite integrals

In view of the Newton-Leibniz's formula (4.3), we can evaluate a Riemann integral $\int_a^b f(x)\,dx$ by finding an anti-derivative of f. This relates the problem of finding area with (the reverse process of) differentiations. Because of this connection, we introduce the notion of **indefinite integrals** which *symbolically* looks like a Riemann integral but *conceptually* different:

Definition 4.6 — Indefinite Integrals. Suppose f is a function defined on an interval I, then the **indefinite integral** of f is defined to be:

$$\int f(x)\,dx := \{F(x) : F'(x) = f(x) \text{ on } I\}.$$

If F_0 is a particular anti-derivative of f, then any other anti-derivative F of f on I would differ from F_0 by a constant, then we also have

$$\int f(x)\,dx := \{F_0(x) + C : C \text{ is a real constant}\}.$$

Usually, we abbreviate the above by $\int f(x)\,dx = F_0(x) + C$ so that students who are not taking honor calculus could understand the notation.

Naturally, $\int_a^b f(x)\,dx$ will then be called a **definite integral** of f. It is computationally similar to the indefinite integral $\int f(x)\,dx$ in view of the Newton-Leibniz formula, but as a math major, you should be very clear about their conceptual difference. You should regards $\int f(x)\,dx$ as

$$\left(\frac{d}{dx}\right)^{-1} f$$

where $\frac{d}{dx}$ is regarded as an operator.

Here are some examples:

$$\begin{aligned}
\frac{d}{dx}\sin x = \cos x &\implies \int \cos x\,dx = \sin x + C \\
\frac{d}{dx}\left(\frac{x^p}{p+1}\right) = x^p \quad \text{where } p \neq -1 &\implies \int x^p\,dx = \frac{x^p}{p+1} + C \\
\frac{d}{dx}\frac{1}{\sqrt{1-x^2}} = \sin^{-1} x &\implies \int \frac{1}{\sqrt{1-x^2}}\,dx + \sin^{-1} x + C
\end{aligned}$$

When writing an indefinite integral, we often implicitly assume that the domain of both f and F is an interval I which is connected. Consider a function f defined on a disjoint union of two intervals:

$$f(x) = \begin{cases} x^3 & \text{if } x \in (0,1) \\ x^4 & \text{if } x \in (2,3) \end{cases}.$$

One anti-derivative of f is certainly

$$F_0(x) = \begin{cases} \frac{x^4}{4} & \text{if } x \in (0,1) \\ \frac{x^5}{5} & \text{if } x \in (2,3) \end{cases},$$

but the others may be of the form

$$F(x) = \begin{cases} \frac{x^4}{4} + C_1 & \text{if } x \in (0,1) \\ \frac{x^5}{5} + C_2 & \text{if } x \in (2,3) \end{cases},$$

where C_1 and C_2 are two real constants, so it is not necessarily of the form $F_0(x) + C$. Therefore it would be **problematic** to say

$$\int f(x)\,dx = F_0(x) + C.$$

When writing

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C,$$

we should implicitly assume the domain involved is an interval not containing 0, such as $(-2,-1)$ or $[1,3)$, but not $(-1,1]$.

The indefinite integral $\int \frac{1}{x}\,dx$ worths some discussion. On the interval $(0,\infty)$, an anti-derivative of $\frac{1}{x}$ is clearly $\log x$, but $\log x$ is undefined if on the interval $(-\infty,0)$. Instead, the anti-derivative of $\frac{1}{x}$ on the interval $(-\infty,0)$ is $\log(-x)$ because by chain rule:

$$\frac{d}{dx}\log(-x) = \frac{d}{d(-x)}\log(-x)\cdot\frac{d(-x)}{dx} = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}.$$

Therefore, we have $\int \frac{1}{x}\,dx = \log x + C$ when the domain interval in the context is a subset of $(0,\infty)$, while $\int \frac{1}{x}\,dx = \log(-x) + C$ when the domain is a subset of $(-\infty,0)$. However, we often write it in a unified way:

$$\int \frac{1}{x}\,dx = \log|x| + C,$$

so that it applies to both interval types. Again, if the domain in the context is an interval like $(-1,1)$, it does not make sense to talk about $\int \frac{1}{x}\,dx$ as the integrand $\frac{1}{x}$ is undefined at 0.

Many of might have known that

$$\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C.$$

Similarly, when writing this we implicitly assume the interval I involved is one that either $\cos x > 0$ on I, or $\cos x < 0$ on I.

We should also be careful when the function f is piecewise defined, such as

$$f(x) = \begin{cases} e^x & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}.$$

It is NOT true that

$$\int f(x)\,dx = \begin{cases} e^x + C & \text{if } x \geq 0 \\ x + C & \text{if } x < 0 \end{cases}, \qquad \text{where } C \text{ is any real constant} \qquad \text{(WRONG!)}$$

or

$$\int f(x)\,dx = \begin{cases} e^x + C_1 & \text{if } x \geq 0 \\ x + C_2 & \text{if } x < 0 \end{cases}, \qquad \text{where } C_1, C_2 \text{ are any real constants} \qquad \text{(WRONG!)}$$

The function

$$F(x) = \begin{cases} e^x + C & \text{if } x \geq 0 \\ x + C & \text{if } x < 0 \end{cases}$$

is not even continuous at 0 as $\lim\limits_{x\to 0+} F(x) = C + 1$ whereas $\lim\limits_{x\to 0^-} F(x) = C$. The same for functions

$$\begin{cases} e^x + C_1 & \text{if } x \geq 0 \\ x + C_2 & \text{if } x < 0 \end{cases}$$

unless C_1 and C_2 are some carefully chosen constants.

In fact, one of the anti-derivative of f should be

$$F_0(x) = \begin{cases} e^x & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases},$$

so we should write

$$\int f(x)\,dx = F_0 + C = \begin{cases} e^x & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases} + C$$

where C is any real constant.

■ **Exercise 4.30** Compute the indefinite integral of the function:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \sin x & \text{if } x < 0 \end{cases}.$$

Also, compute $\int |x|\,dx$ (take the domain to be $\mathbb{R}$)

Analogous results of (2) and (3) in Proposition 4.10 for Riemann (i.e. definite) integrals also hold for indefinite integral, such as

$$\int_c f(x)\,dx = c\int f(x)\,dx \quad \text{and} \quad \int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

The proof is much easier. To prove the second statement, we take anti-derivatives F of f, and G of g. Then, we have

$$\int f(x)\,dx + \int g(x)\,dx = F(x) + C_1 + G(x) + C_2$$

where C_1, C_2 are any real constants. Since $(F+G)' = f + g$ by the linearity of differentiations, $F + G$ is an anti-derivative of $f + g$ and so

$$\int \big(f(x) + g(x)\big)\,dx = F(x) + G(x) + C_3$$

where C_3 is any real constant. We are only left to show

$$\{C_1 + C_2 : C_1, C_2 \in \mathbb{R}\} = \{C_3 : C_3 \in \mathbb{R}\}$$

which is trivial (just prove both $\subset$ and $\supset$).