4 — Integrations

4.1 Jordan Measure

You may have learned in high school that a definite integral of a function such as

$$\int_0^1 x^3\,dx$$

means the area under the graph $y = x^3$ from $x = 0$ to $x = 1$, or more precisely, the area of the region

$$\{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le x^3\}.$$

You were probably told (but not explained) that a definite integral like this can be evaluated by finding an anti-derivative of the function, namely $\frac{d}{dx}\frac{x^4}{4} = x^3$, so we have

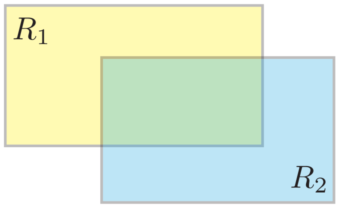
$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}.$$

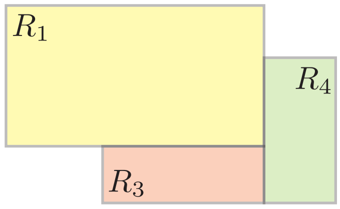
This is known as the Fundamental Theorem of Calculus, or Newton-Leibniz formula that relates the problem of finding area and differentiations.

We all learned about the concept of area in primary school (if not earlier), and know that the area of a triangle is a half of the base times height, and the area of a circle with radius r is πr^2. However, it seems like we never learned about the rigorous definition of *area*! What is meant by the area of a region? In this section, we introduce one definition of area, the Jordan measure – which is will be used to give the rigorous definition of Riemann integrals.

4.1.1 Simple regions

Before introducing the Jordan measure of an arbitrary region in $\mathbb{R}^2$, we first focus on some *simple* regions. We first declare that the area of a rectangle $\langle a, b\rangle \times \langle c, d\rangle \subset \mathbb{R}^2$ to be $(b-a)(d-c)$. Here $\langle$ means (or [, and $\rangle$ means) or]. Intuitively, the intersection $R_1 \cap R_2$ of two overlapping rectangles R_1 and R_2 is also a rectangle, whereas the union $R_1 \cup R_2$ needs not to be a rectangle. However, one can break down $R_1 \cup R_2$ into three smaller rectangles $R_1 \cup R_3 \cup R_4$ so that these three rectangles are disjoint.





It is then sensible to define the area of $R_1 \cup R_2$ to be the sum of areas of R_1, R_3 and R_4. Inductively, one can see if we have finitely many rectangles, their union can be decomposed into smaller rectangles which are disjoint, and hence we can define the area of such a union. To summarize, we have:

Theorem 4.1 — Simple Regions. A set $E \subset \mathbb{R}^2$ is called a **simple region** if E is the union of finitely many rectangles, i.e.

$$E = \bigcup_{i=1}^{N} \langle a_i, b_i \rangle \times \langle c_i, d_i \rangle,$$

where $a_i \leq b_i$ and $c_i \leq d_i$, and they are all finite real numbers for any i.

We allow $a_i = b_i$ or $c_i = d_i$ in the above definition. In other words, we also count line segments and points to be "rectangles".

Proposition 4.2 Every simple region E can be expressed as the union of finitely many rectangles in $\mathbb{R}^2$ which are mutually disjoint.

Proof. By induction and the law that $(\coprod_i A_i) \cup B = \coprod_i (A_i \cup B)$. ■

Definition 4.1 — Area of Simple Regions. Let $E \subset \mathbb{R}^2$ be a simple region so that, in view of the above proposition, can be expressed as

$$E = \coprod_{i=1}^{N} \langle a_i, b_i \rangle \times \langle c_i, d_i \rangle$$

where $\big(\langle a_i, b_i \rangle \times \langle c_i, d_i \rangle\big) \cap \big(\langle a_j, b_j \rangle \times \langle c_j, d_j \rangle\big) = \emptyset$ wherever $i \neq j$. We define the area of $E \subset \mathbb{R}^2$ to be:

$$A(E) := \sum_{i=1}^{N} (b_i - a_i)(d_i - c_i).$$

A simple region E can be expressed as the union of disjoint rectangles in many different ways! It is possible to prove the definition of $A(E)$ is independent of how we express E as the union of disjoint rectangles. Again, it is an intuitive fact which cannot be easily proved.

4.1.2 General region in $\mathbb{R}^2$

Now consider a general region Ω in $\mathbb{R}^2$. In primary school, we probably have learned how to find its approximate area by counting the number of little squares in a grid. The rigorous definition of area of Ω is in fact motivated by the idea of counting squares. We approximate Ω by simple regions from inside and also from outside (such as S and T in Figure 4.1).

As S and T are simple regions, it makes sense to talk about their areas $A(S)$ and $A(T)$. It is then sensible to expect that the *area* of Ω should be bounded between $A(S)$ and $A(T)$. As we approximate Ω by a pair of "closer", more "refined" simple regions S and T , we expect the "area"

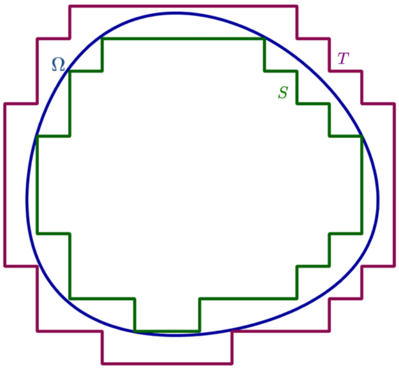


Figure 4.1: Approximation of Ω by simple regions S and T

of Ω should be "something like" $\lim\limits_{S\to\Omega^-} A(S)$ and $\lim\limits_{T\to\Omega^+} A(T)$. We learned about what is the limit of a sequences or a function, but we never learn about limit of sets.

Instead of using "limits", we define the area of Ω by taking it to be the "best" approximated area by simple regions. If we approximate Ω by simple regions from inside (such as S), then the "best" means the maximum possible $A(S)$ among all possible simple regions $S \subset \Omega$. Similarly, if we approximate Ω by simple regions from outside (such as T), then the "best" means the minimum possible $A(T)$ among all simple regions $T \supset \Omega$. That is exactly what Jordan measure means. Precisely, we have:

Definition 4.2 — Jordan Measure in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded set. We define its **inner Jordan measure** $\mu_*(\Omega)$ and **outer Jordan measure** $\mu^*(\Omega)$ to be:

$$\mu_*(\Omega) := \sup\{A(S) : S \subset \Omega \text{ and } S \text{ is a simple region in } \mathbb{R}^2\}$$
$$\mu^*(\Omega) := \inf\{A(T) : T \supset \Omega \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}$$

(Reference: Figure 4.2).

If $\mu_*(\Omega) = \mu^*(\Omega)$, then we say Ω is **Jordan measurable**, and define its **Jordan measure** $\mu(\Omega)$ by taking $\mu(\Omega) := \mu^*(\Omega)$.

Whenever S and T are simple regions such that $S \subset \Omega \subset T$, it is intuitively clear (but not easy to prove) that $A(S) \leq A(T)$. Therefore, one must have $\mu_*(\Omega) \leq \mu^*(\Omega)$.

Clearly, if $\mu^*(\Omega) = 0$, then Ω must be Jordan measurable and $\mu(\Omega) = 0$.

Let's first get a sense of Jordan measure by some elementary examples:

Proposition 4.3 Any simple region E in $\mathbb{R}^2$ is Jordan measurable, and $\mu(E) = A(E)$.

Proof. Since E is a simple region, E satisfies the criterion:

$$E \subset E \text{ and } E \text{ is a simple region in } \mathbb{R}^2$$

in the definition of $\mu_*(E)$. Therefore, $A(E)$ belongs to the set:

$$\{A(S) : S \subset E \text{ and } S \text{ is a simple region in } \mathbb{R}^2\}.$$

An element belonging to a set must be bounded above by its supremum (which is one of its upper bound), so we have:

$$A(E) \leq \sup \underbrace{\{A(S) : S \subset E \text{ and } S \text{ is a simple region in } \mathbb{R}^2\}}_{A(E) \text{ belongs to this set}} = \mu_*(E).$$

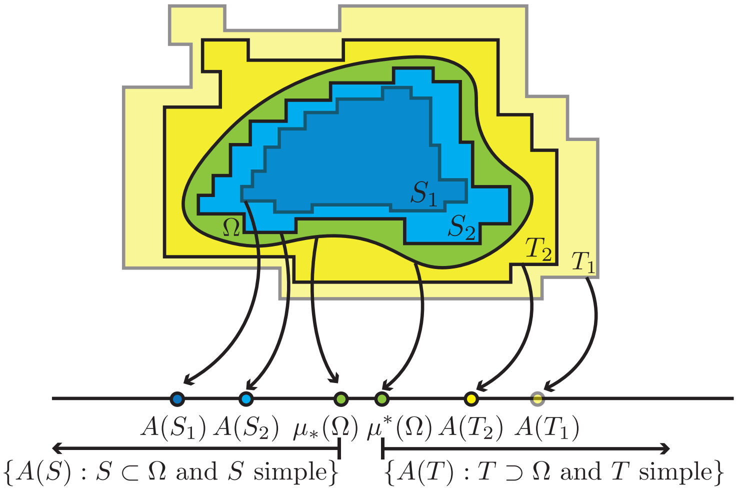


Figure 4.2: Outer and Inner Jordan Measures

The same argument, *mutatis mutandis*, proves that

$$A(E) \geq \inf \underbrace{\{A(T) : T \supset E \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}}_{A(E) \text{ belongs to this set}} = \mu^*(E).$$

In conclusion, we have proved:

$$A(E) \leq \mu_*(E) \leq \mu^*(E) \leq A(E).$$

Therefore, the only possible is they are all equal to each other. It proves E is Jordan measurable and $\mu(E) = \mu_*(E) = A(E)$. ■

■ **Exercise 4.1** Let X and Y be bounded sets in $\mathbb{R}$ such that $X \subset Y$. Show that $\sup X \leq \sup Y$ and $\inf X \geq \inf Y$. Hence, show that if Ω_1, Ω_2 are bounded regions in $\mathbb{R}^2$ such that $\Omega_1 \subset \Omega_2$, then we have $\mu_*(\Omega_1) \leq \mu_*(\Omega_2)$ and $\mu^*(\Omega_1) \leq \mu^*(\Omega_2)$.

The Jordan measure of a simple region can be found directly from the definition. For a general region Ω, we typically calculate its Jordan measure by taking a pair of sequences $\{S_n\}$ and $\{T_n\}$ of simple regions, with $S_n \subset \Omega$ and $T_n \supset \Omega$ for any n, such that $\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty} A(T_n)$. Let's look the example of a parallelogram:

■ **Example 4.1** Consider the parallelogram Ω with vertices

$$O(0,0), A(b,0), B(b+c,h), C(c,h)$$

where $b, c, h > 0$, i.e. the parallelogram has base length b and height h. Show that Ω is Jordan measurable and $\mu(\Omega) = bh$.

■ **Solution** For each $n \in \mathbb{N}$, we define the simple region:

$$T_n := \bigcup_{k=1}^{n} \left[\frac{(k-1)c}{n}, \frac{kc}{n} + b\right] \times \left[\frac{(k-1)h}{n}, \frac{kh}{n}\right].$$

See Figure 4.3 for the illustration.

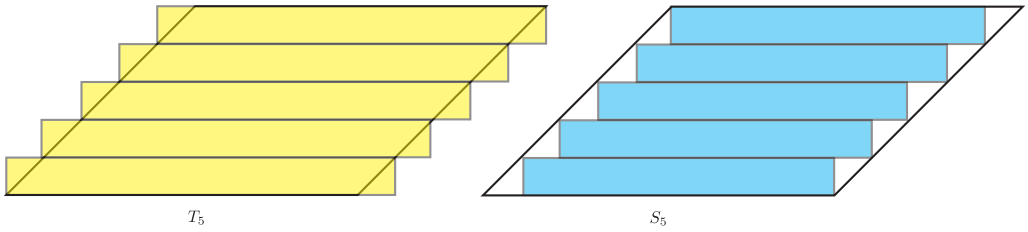


Figure 4.3: Parallelogram and its simple region approximations

The simple region T_n contains the parallelogram Ω, so by the definition of outer Jordan measure (which is the infimum of all area of outer simple regions), we have

$$\mu^*(\Omega) \leq A(T_n) = n \cdot \left(b + \frac{c}{n}\right) \cdot \frac{h}{n} = h\left(b + \frac{c}{n}\right).$$

Similarly, one can also construct inner simple region S_n:

$$S_n := \bigcup_{k=1}^{n} \left[\frac{kc}{n}, \frac{(k-1)c}{n} + b\right] \times \left[\frac{(k-1)h}{n}, \frac{kh}{n}\right].$$

Then, we have

$$\mu_*(\Omega) \geq A(S_n) = n \cdot \left(b - \frac{c}{n}\right) \cdot \frac{h}{n} = h\left(b - \frac{c}{n}\right).$$

In summary, we have proved that for each $n \in \mathbb{N}$,

$$h\left(b - \frac{c}{n}\right) \leq \mu_*(\Omega) \leq \mu^*(\Omega) \leq h\left(b + \frac{c}{n}\right).$$

Letting $n \to \infty$, we conclude that:

$$hb \leq \mu_*(\Omega) \leq \mu^*(\Omega) \leq hb,$$

and therefore $\mu_*(\Omega) = \mu^*(\Omega) = hb$, so Ω is Jordan measurable and $\mu(\Omega) = hb$.

■ **Exercise 4.2** Show that any straight line segment has Jordan measure zero. Note that a straight line may not be horizontal or vertical.

■ **Exercise 4.3** Show that any right-angled triangle with one side vertical and one side horizontal is Jordan measurable and its Jordan measure is given by $\frac{1}{2} \times$ base $\times$ height.

In general, if there exist sequences of inner simple regions $\{S_n\}$ and outer simple regions $\{T_n\}$ such that $A(T_n)$ and $A(S_n)$ converge to the same limit, we can conclude that the region is Jordan measure. In fact, the converse is also true. Let's state it as a proposition:

Proposition 4.4 Let Ω be a non-empty bounded region in $\mathbb{R}^2$. Then the following are equivalent:

1. there exist sequences of inner simple regions $\{S_n\}$ and outer simple regions $\{T_n\}$ of Ω such that $\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty} A(T_n) = m$.
2. Ω is Jordan measurable and $\mu(\Omega) = m$.

Proof. For (1) $\implies$ (2), the proof is similar to the parallelogram example. By the definition of

μ^* and μ_*, we have

$$A(S_n) \le \mu_*(\Omega) \le \mu^*(\Omega) \le A(T_n) \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ and by $\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty} A(T_n) = m$, we conclude that $\mu_*(\Omega) = \mu^*(\Omega) = m$.

For (2) $\implies$ (1), we recall that the definition of $\mu^*(\Omega)$ is given by

$$\mu^*(\Omega) = \inf\{A(T) : T \supset \Omega \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}$$

"Infimum" means the greatest lower bound, so for any $n \in \mathbb{N}$, there must exist a simple region $T_n \supset \Omega$ such that

$$\mu^*(\Omega) \le A(T_n) < \mu^*(\Omega) + \frac{1}{n},$$

otherwise $\mu^*(\Omega) + \frac{1}{n}$ would also be a lower bound of the set

$$\{A(T) : T \supset \Omega \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}.$$

By squeeze theorem, it is clear that

$$\lim_{n\to\infty} A(T_n) = \mu^*(\Omega).$$

Similarly, for each $n \in \mathbb{N}$, there exists a simple region $S_n \subset \Omega$ such that

$$\mu_*(\Omega) - \frac{1}{n} < A(S_n) \le \mu_*(\Omega).$$

Letting $n \to \infty$, we also have

$$\lim_{n\to\infty} A(S_n) = \mu_*(\Omega).$$

Since Ω is Jordan measurable, we have

$$\lim_{n\to\infty} A(T_n) = \mu^*(\Omega) = m = \mu_*(\Omega) = \lim_{n\to\infty} A(S_n).$$

■

■ **Exercise 4.4** Prove using Proposition 4.4 that the trapezium Ω with vertices:

$$O(0,0), A(a,0), B(b+c,h), C(c,h)$$

where $a,b,c,h > 0$, is Jordan measurable and $\mu(\Omega) = \frac{1}{2}(a+b)h$.

■ **Exercise 4.5** Prove that for a non-empty bounded region Ω in $\mathbb{R}^2$, the following is also equivalent to (1) and (2) in Proposition 4.4:

"$\forall \varepsilon > 0$, $\exists$ simple regions $S \subset \Omega$ and $T \supset \Omega$ such that $A(T \backslash S) < \varepsilon$."

[Hint: You can use the fact that if $X \subset Y \subset Z$ are all simple regions, then $Z\backslash Y$, $Z\backslash X$ and $Y\backslash X$ are simple regions too, with $A(Z\backslash Y) \le A(Z\backslash X)$ and $A(Y\backslash X) \le A(Z\backslash X)$.]

4.1.3 Finite additivity and isometric invariance

Next we prove two good properties about Jordan measure:

- If Ω_1 and Ω_2 are two disjoint bounded Jordan measurable regions in $\mathbb{R}^2$, then so is $\Omega_1 \cup \Omega_2$ and $\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2)$.
- If $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a distance-preserving map (e.g. rotations, reflections, translations and their compositions), then $\mu_*(\Phi(\Omega)) = \mu_*(\Omega)$ and $\mu^*(\Phi(\Omega)) = \mu^*(\Omega)$ for any bounded region Ω in $\mathbb{R}^2$.

These sound intuitive, but it is not very obvious from the definition of Jordan measures (which involve sup and inf).

Proposition 4.5 — Finite Additivity. Suppose Ω_1 and Ω_2 are two bounded Jordan measurable regions in $\mathbb{R}^2$, then so is $\Omega_1 \cup \Omega_2$. Furthermore, if $\Omega_1 \cap \Omega_2 = \emptyset$, then $\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2)$.

Proof. We use Proposition 4.4. Given that Ω_1 and Ω_2 are Jordan measurable, there exist sequences of inner simple regions $\{S_n^{(i)}\}_{n=1}^{\infty}$ and outer simple regions $\{T_n^{(i)}\}_{n=1}^{\infty}$, where $i = 1, 2$, such that $S_n^{(i)} \subset \Omega_i \subset T_n^{(i)}$ for each $n \in \mathbb{N}$ and $i \in \{1, 2\}$, and

$$\lim_{n\to\infty} A\big(S_n^{(i)}\big) = \lim_{n\to\infty} A\big(T_n^{(i)}\big) = \mu(\Omega_i).$$

From elementary set theory, we know

$$\big(T_n^{(1)} \cup T_n^{(2)}\big)\backslash\big(S_n^{(1)} \cup S_n^{(2)}\big) \subset \big(T_n^{(1)}\backslash S_n^{(1)}\big) \cup \big(T_n^{(2)}\backslash S_n^{(2)}\big).$$

Therefore, we get

$$A\Big(\big(T_n^{(1)} \cup T_n^{(2)}\big)\backslash\big(S_n^{(1)} \cup S_n^{(2)}\big)\Big) \le A\Big(\big(T_n^{(1)}\backslash S_n^{(1)}\big)\Big) + A\Big(\big(T_n^{(2)}\backslash S_n^{(2)}\big)\Big).$$

For two simple regions $X \subset Y$, it is intuitive (yet tricky to prove) that $0 \le A(Y\backslash X) = A(Y) - A(X)$, so it follows that

$$0 \le A\big(T_n^{(1)} \cup T_n^{(2)}\big) - A\big(S_n^{(1)} \cup S_n^{(2)}\big) \le \underbrace{A\big(T_n^{(1)}\big) - A\big(S_n^{(1)}\big)}_{\to 0} + \underbrace{A\big(T_n^{(2)}\big) - A\big(S_n^{(2)}\big)}_{\to 0}.$$

By squeeze theorem, we conclude that

$$\lim_{n\to\infty} A\big(T_n^{(1)} \cup T_n^{(2)}\big) = \lim_{n\to\infty} A\big(S_n^{(1)} \cup S_n^{(2)}\big).$$

Noting that $T_n^{(1)} \cup T_n^{(2)}$ and $S_n^{(1)} \cup S_n^{(2)}$ are simple regions such that $S_n^{(1)} \cup S_n^{(2)} \subset \Omega_1 \cup \Omega_2 \subset T_n^{(1)} \cup T_n^{(2)}$, we conclude by Proposition 4.4 that $\Omega_1 \cup \Omega_2$ is Jordan measurable with

$$\mu(\Omega_1 \cup \Omega_2) = \lim_{n\to\infty} A\big(S_n^{(1)} \cup S_n^{(2)}\big).$$

To find $\mu(\Omega_1 \cup \Omega_2)$ when $\Omega_1 \cap \Omega_2 = \emptyset$, we observe that $S_n^{(1)} \cap S_n^{(2)} = \emptyset$ for any n (while it is not true for the outer simple regions). Therefore,

$$\mu(\Omega_1 \cup \Omega_2) = \lim_{n\to\infty} A\big(S_n^{(1)} \cup S_n^{(2)}\big) = \lim_{n\to\infty} \Big(A\big(S_n^{(1)}\big) + A\big(S_n^{(2)}\big)\Big) = \mu(\Omega_1) + \mu(\Omega_2).$$

■

■ **Exercise 4.6** Prove by induction that for any **finitely** many bounded Jordan measurable regions $\Omega_1, \cdots, \Omega_N$ in $\mathbb{R}^2$, then the union $\Omega_1 \cup \cdots \cup \Omega_N$ is also Jordan measurable. Also, if $\Omega_i \cap \Omega_j = \emptyset$ for any $i \neq j$, then

$$\mu(\Omega_1 \cup \cdots \cup \Omega_N) = \mu(\Omega_1) + \cdots + \mu(\Omega_N).$$

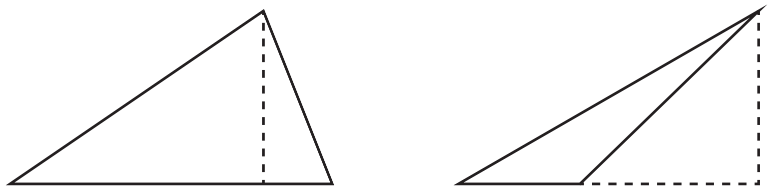
■ **Exercise 4.7** Prove that if Ω_1 and Ω_2 are two bounded Jordan measurable regions in $\mathbb{R}^2$, then so are $\Omega_1 \cap \Omega_2$ and $\Omega_1 \backslash \Omega_2$.

1. Prove that $\mu(\Omega_1 \cup \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2) - \mu(\Omega_1 \cap \Omega_2)$.
2. Assume further that $\Omega_1 \subset \Omega_2$, prove that $\mu(\Omega_2 \backslash \Omega_1) = \mu(\Omega_2) - \mu(\Omega_1)$.

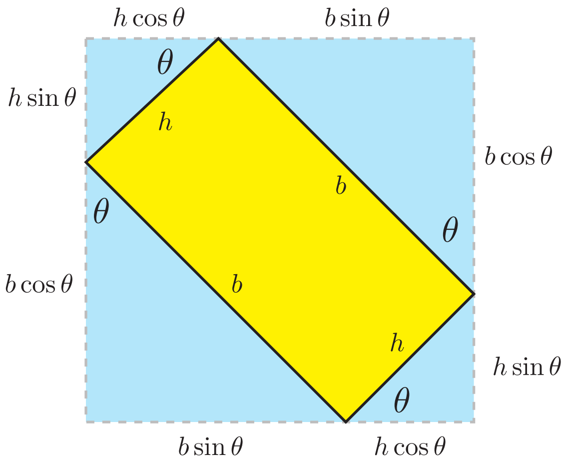
In Exercise 4.3, we proved that the area of a triangle with its base being horizontal and height being vertical is given by:

$$\frac{1}{2} \times \text{base} \times \text{height}.$$

Then, how about a general triangle? Using finite additivity, this formula can be extended to triangles with one of the side being horizontal by considering the following diagrams:



For a general triangle, we need to prove that the Jordan measure is invariant under isometries (a.k.a. distance-preserving maps) such as rotations, reflections, translations. It suffices to prove the measure of a rectangle is invariant under isometries. According to the diagram below, finite additivity of Jordan measures and Exercise 4.3, one can show that the measure of *any* rectangle is always base times height. The base and height are preserved under isometry, so is the measure of the rectangle.



$$\begin{aligned}&\text{Jordan measure of the yellow rectangle}\\&= \underbrace{(h\cos\theta + b\sin\theta)(h\sin\theta + b\cos\theta)}_{\text{Jordan measure of the blue rectangle}} - 2\cdot\frac{1}{2}h^2\sin\theta\cos\theta\\&\qquad - 2\cdot\frac{1}{2}b^2\sin\theta\cos\theta = hb(\cos^2\theta + \sin^2\theta) = hb.\end{aligned}$$

Now the invariance under isometry of Jordan measures can be extended to general Jordan measurable regions in $\mathbb{R}^2$:

> **Proposition 4.6 — Isometric Invariance.** Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a distance-preserving map, and Ω be a bounded Jordan measurable region in $\mathbb{R}^2$. Then $\Phi(\Omega)$ is also Jordan measurable and $\mu(\Phi(\Omega)) = \mu(\Omega)$.

Proof. As per the above discussion, the measure of a rectangle is preserved under such a map Φ, so the measure of simple regions (which are disjoint unions of rectangles) is also preserved too.

Take a sequence of outer simple regions $\{T_n\}$ with $\Omega \subset T_n$ for any n, and $\lim_{n\to\infty} A(T_n) = \mu^*(\Omega)$. Then, by $\Phi(\Omega) \subset \Phi(T_n)$, we have from Exercise 4.1

$$\mu^*(\Phi(\Omega)) \leq \mu^*(\Phi(T_n)) = \mu^*(T_n) = A(T_n).$$

Similarly, take a sequence $\{S_n\}$ of inner simple regions (i.e. $S_n \subset \Omega$) such that $A(S_n) \to \mu_*(\Omega)$ as $n \to \infty$. By Exercise 4.1 and $\Phi(S_n) \subset \Phi(\Omega)$, we also have

$$A(S_n) = \mu_*(\Phi(S_n)) \leq \mu_*(\Phi(\Omega)).$$

To summarize, we have for any $n \in \mathbb{N}$ that

$$A(S_n) \leq \mu_*(\Phi(\Omega)) \leq \mu^*(\Phi(\Omega)) \leq A(T_n) \ \ \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we proved:

$$\mu_*(\Omega) \leq \mu_*(\Phi(\Omega)) \leq \mu^*(\Phi(\Omega)) \leq \mu^*(\Omega).$$

Since Ω is Jordan measurable, the above is in fact an equality. This proves our desired results. ■

Using finite additivity and isometric invariance, we can find the Jordan measure of polygon figures by splitting it into disjoint triangles, rectangles, etc.

■ **Exercise 4.8** Suppose $\Omega \subset \mathbb{R}^2$ is a bounded region such that there exist sequences $\{E_n\}$ and $\{F_n\}$ of bounded Jordan measurable sets with $E_n \subset \Omega \subset F_n$ for any n, and

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(F_n) = m.$$

Show that Ω is also Jordan measurable and $\mu(\Omega) = m$.

■ **Exercise 4.9** Using the previous exercises, show that a circle with radius r is Jordan measurable and has measure πr^2. [Hint: take E_n's and F_n's to be regular polygons.]

4.1.4 Examples of non-measurable sets

There does exist some "strange" sets in $\mathbb{R}^2$ which are not Jordan measurable. Here is one example:

$$\Omega := \{(x, y) : x, y \in \mathbb{Q}, 0 \leq x, y \leq 1\}.$$

Any inner simple region S contained inside Ω must be a finite set of points, since the only "rectangles" contained inside Ω are single points. This shows $\mu_*(\Omega) = 0$.

However, for any outer simple region T containing Ω, we claim that the closure $\overline{T}$ (i.e. the union of T and its boundary) contains $[0,1] \times [0,1]$. It is by the density of rational numbers. For any $(a, b) \in [0,1] \times [0,1]$ one can take a sequence $x_n \in \mathbb{Q} \to a$ and $y_n \in \mathbb{Q} \to b$ as $n \to \infty$. Then $(x_n, y_n) \in \Omega \subset T$ for any n. By order rule, the limit (a, b) of the points (x_n, y_n) must be in T or on its boundary. Therefore, $[0,1] \times [0,1] \subset \overline{T}$, so $A(T) = A(\overline{T}) \geq 1$. This concludes that

$$\mu^*(\Omega) = \inf \underbrace{\{A(T) : T \supset \Omega \text{ and } T \text{ is a simple region in } \mathbb{R}^2\}}_{\text{all elements} \geq 1} \geq 1.$$

Therefore, $\mu_*(\Omega) \neq \mu^*(\Omega)$. The set Ω is not Jordan measurable.

■ **Exercise 4.10** Show that $(\mathbb{Q} \cap [0,1]) \times [0,1]$ is not Jordan measurable.

In MATH 3033/3043, we will study an even more important type of measure called Lebesgue measure, which is an improved version of measure that makes these rational sets to be measurable. The Lebesgue measure also enjoys an even better additivity called *countable additivity*.