4.2 Riemann Integrals

Given that we have defined rigorously the meaning of area in the previous section, we are now ready to introduce the definition of Riemann integrals. We will use Jordan measure. This idea is not originally from Riemann, but by Orrin Frink who related Jordan measure and Riemann integrals together in his paper[1] published in 1933.

Here we will exclusively discuss **bounded** function defined on a closed and **bounded** interval $[a, b]$. If either one of the boundedness conditions is removed, the integral will be called an improper integral which will be discussed later. Given such a function $f : [a, b] \to \mathbb{R}$, we define

$$
\begin{aligned}
G^+_{[a,b]}(f) &:= \{(x, y) : a \leq x \leq b,\ f(x) \geq 0,\ \text{ and } \ 0 \leq y \leq f(x)\} \\
G^-_{[a,b]}(f) &:= \{(x, y) : a \leq x \leq b,\ f(x) < 0,\ \text{ and } \ f(x) \leq y \leq 0\} \\
G_{[a,b]}(f) &:= G^+_{[a,b]}(f) \cup G^-_{[a,b]}(f)
\end{aligned}
$$

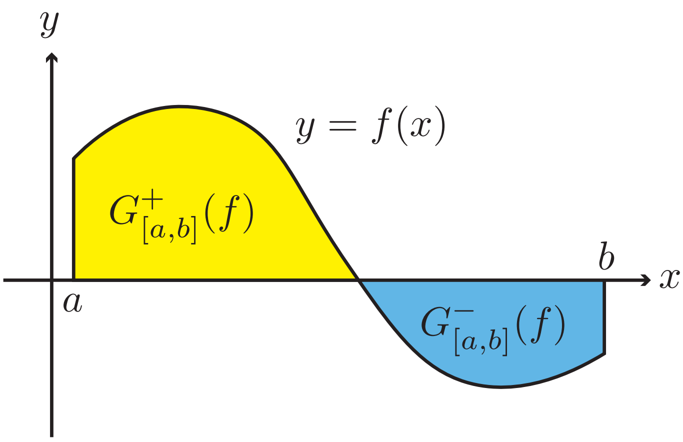


Figure 4.4: $G^+_{[a,b]}(f)$ and $G^-_{[a,b]}(f)$.

■ **Exercise 4.11** Show that the following are equivalent:

1. Both $G^+_{[a,b]}(f)$ and $G^-_{[a,b]}(f)$ are Jordan measurable.
2. $G_{[a,b]}(f)$ is Jordan measurable.

Definition 4.3 — Riemann Integrals. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, then we say f is **Riemann integrable** on $[a, b]$ if and only if $G_{[a,b]}(f)$ is Jordan measurable. In this case, we define

$$\int_a^b f(x)\, dx = \mu\big(G^+_{[a,b]}(f)\big) - \mu\big(G^-_{[a,b]}(f)\big).$$

See Figure 4.4.

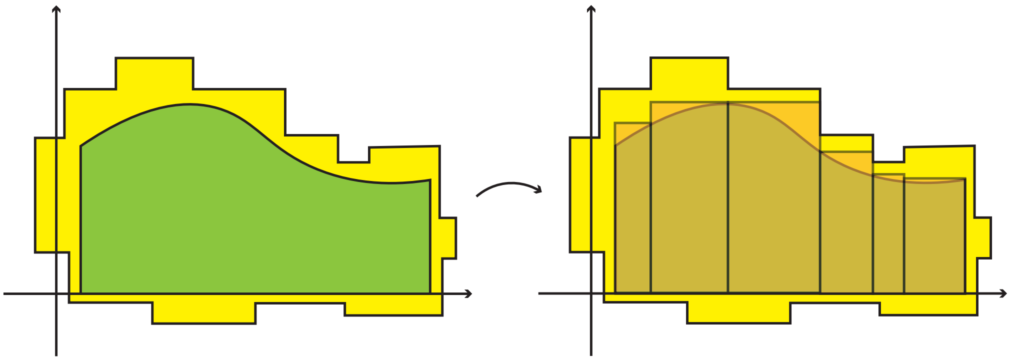
ⓘ We assign negative value to the part below the x-axis. One reason for doing so is to guarantee that if $f(x) \leq g(x) \leq 0$ on $[a, b]$, we still have $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

[1] Orrin Frink, *Jordan Measure and Riemann Integration*, Annals of Mathematics, Second Series, Vol. 34, No. 3 (July 1933), pp. 518-526

4.2.1 Non-negative functions

Let's discuss how we could determine whether $G^+_{[a,b]}(f)$ and $G^-_{[a,b]}(f)$ are Jordan measurable. For simplicity, we label them by G^+ and G^- respectively.

We first consider bounded functions which are **non-negative**, so that we could only consider G^+. We need to consider the outer simple regions and inner simple regions of G^+. One nice property about a region like G^+ is that any outer simple region containing G^+ can be shrunk to become a "bar chart" type region like below with each vertical bar barely hit the graph $y = f(x)$:

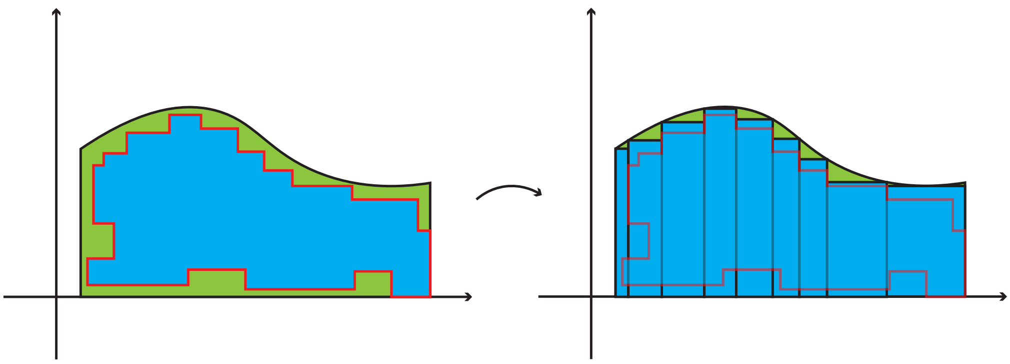


Recall that the outer measure of G^+ is defined as

$$\mu^*(G^+) := \inf\{A(T) : G^+ \subset T \text{ and } T \text{ is simple}\}.$$

For each simple region T containing G^+ there is always a smaller "bar chart" region T' (to be more precisely defined soon) with $G^+ \subset T' \subset T$, so the above infimum can be taken over all "bar chart" regions containing G^+ only. It is because dropping those non-bar-chart simple regions will not affect the infimum. Here is an analogy: say in a test, you know that your test score is higher than one of your friend, then you know that the lowest is not you!

Likewise, any inner simple region contained in G^+ can also be expanded to become an inner "bar chart" region like below:



The inner measure of G^+ is defined as

$$\mu_*(G^+) := \sup\{A(S) : S \subset G^+ \text{ and } S \text{ is simple}\}.$$

By similar rationale as the outer measure, one can simply take the supremum over all "bar chart" regions contained in G^+ only.

To describe these "bar chart" regions in a more precise way, we can define a **partition** of $[a, b]$:

$$P : a := x_0 < x_1 < \cdots < x_n =: b,$$

and its associated outer and inner "bar chart" regions are respectively

$$T_P := \bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [0, M_i] \qquad \text{where } M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$S_P := \bigcup_{i=1}^{n} [x_{i-1}, x_i] \times [0, m_i) \qquad \text{where } m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

The areas of T_P and S_P are often called respectively the **upper Darboux sum** and **lower Darboux sum** of f with respect to partition P, which are denoted by:

$$U(P, f) := A(T_P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$L(P, f) := A(S_P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

As discussed, the outer and inner measures of G^+ can be defined by taking the sup and inf over all "bar chart" regions (which can be described using partitions of $[a, b]$), so we have:

$$\mu^*(G^+) = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\},$$
$$\mu_*(G^+) = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}.$$

Definition 4.4 — Upper and Lower Darboux Integrals. Let $f : [a, b] \to \mathbb{R}$ be a bounded function (not necessarily non-negative), we define and denote the upper and lower Darboux integrals as:

$$\overline{\int_a^b} f(x)\, dx := \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$$
$$\underline{\int_a^b} f(x)\, dx := \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$$

The upper and lower Darboux integrals can be defined on any bounded function f on $[a, b]$, not only on non-negative functions. But if f is non-negative and bounded on $[a, b]$, we then have:

$$\overline{\int_a^b} f(x)\, dx = \mu^*(G^+) \qquad\qquad \underline{\int_a^b} f(x)\, dx = \mu_*(G^+).$$

■ **Example 4.2** Show that $f(x) = x^2$ is Riemann integrable on $[0, 1]$ and find $\displaystyle\int_0^1 x^2\, dx$ from the definition.

■ **Solution** Similar to proving a region in $\mathbb{R}^2$ is Jordan measurable, we will construct a sequence of partitions P_n of $[0, 1]$ such that $U(P_n, f)$ and $L(P_n, f)$ converge to the same limit.

For each $n \in \mathbb{N}$, consider the partition

$$P_n : x_0 := 0 < \underbrace{\frac{1}{n}}_{x_1} < \underbrace{\frac{2}{n}}_{x_2} < \cdots < \underbrace{\frac{n-1}{n}}_{x_{n-1}} < 1 =: x_n.$$

Then, for any $i \in \{1, \cdots, n\}$, we have

$$M_i := \sup_{x \in [\frac{i-1}{n}, \frac{i}{n}]} x^2 = \frac{i^2}{n^2},$$

$$m_i := \inf_{x \in [\frac{i-1}{n}, \frac{i}{n}]} x^2 = \frac{(i-1)^2}{n^2}.$$

Hence,

$$U(P_n, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n},$$

$$L(P_n, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=0}^{n-1} \frac{i^2}{n^3} = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} = \frac{1}{6} \cdot \frac{n-1}{n} \cdot \frac{2n-1}{n}.$$

It is easy to see that

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \frac{1}{3}.$$

By applying squeeze theorem on the inequality:

$$L(P_n, x^2) \leq \underline{\int_0^1} x^2\, dx = \mu_*(G^+) \leq \mu^*(G^+) = \overline{\int_0^1} x^2\, dx \leq U(P_n, x^2), \quad \forall n \in \mathbb{N},$$

we get

$$\underline{\int_0^1} x^2\, dx = \overline{\int_0^1} x^2\, dx = \frac{1}{3}.$$

Hence, x^2 is integrable on $[0, 1]$ and we have

$$\int_0^1 x^2\, dx = \frac{1}{3}.$$

■ **Exercise 4.12** Let $f : [a, b] \to \mathbb{R}$ be a non-negative[a], bounded function. Suppose there exists a sequence of partitions P_n of $[a, b]$ such that

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = I,$$

then f is Riemann integrable on $[a, b]$ and $\displaystyle\int_a^b f(x)\, dx = I$.

[a]We will extend the result to any bounded function later.

■ **Exercise 4.13** Show that e^x is Riemann integrable on any closed and bounded interval $[a, b]$, and find $\displaystyle\int_a^b e^x\, dx$.

■ **Exercise 4.14** The following classic formula was discovered by Jacob Bernoulli in 1713:

$$1^p + 2^p + \cdots + n^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j C_j^{p+1} B_j n^{p+1-j}, \qquad p \in \mathbb{N}$$

where B_j's are so-called *Bernoulli's numbers* given by:

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \cdots$$

The proof of the above formula can be found in some standard number theory or complex analysis textbooks. Using this formula without proof, show that x^p (where $p \in \mathbb{N}$) is Riemann integrable on $[0,1]$ and that:

$$\int_0^1 x^p \, dx = \frac{1}{p+1}, \quad \text{where } p \text{ is a positive integer}$$

from the definition of integrals.

■ **Exercise 4.15** First prove the formula:

$$2\sin\frac{x}{2} \cdot (\sin x + \sin 2x + \cdots + \sin nx) = \cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, show that $\sin x$ is Riemann integrable on $[0,\pi]$, and find the value of

$$\int_0^\pi \sin x \, dx.$$

■ **Example 4.3** Consider the function $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ in the most simplified form } (m, n \in \mathbb{N}) \end{cases}$$

For instance, we have $f(\frac{2}{3}) = \frac{1}{3}$, $f(\frac{8}{14}) = f(\frac{4}{7}) = \frac{1}{7}$. Show that f is Riemann integrable on $[0,1]$ and $\displaystyle\int_0^1 f(x)\,dx = 0$.

■ **Solution** Let P_n be the partition $0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1$ where $n \geq 5$ is prime. For any $i = 0, 1, \ldots, n-1$, the interval $[i/n, (i+1)/n]$ must contain at least one irrational number, so we must have:

$$\inf_{[i/n,(i+1)/n]} f = 0.$$

This immediately shows $L(P_n, f) = 0$.

Next we estimate $U(P_n, f)$ from above. The rational numbers r in $[0,1]$ that give the largest output $f(r)$ are given in descending order by:

$$\{r_j\}_{j=1}^\infty = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\right\}$$

as the outputs are:

$$\{f(r_j)\}_{j=1}^\infty = \left\{1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \cdots\right\}$$

The worst scenario for $U(P_n, f)$ is there is exactly one of $\{r_1, r_2, \cdots, r_n\}$ in each $[i/n, (i+1)/n]$,

$i = 0, 1, \cdots, n-1$, and so

$$U(P_n, f) \le \frac{1}{n}(f(r_1) + \cdots + f(r_n)).$$

Here is why we want n to be a prime: note that when $n \ge 5$ is a prime, it is impossible for $\frac{i}{n} = r_j$ for any $i = 0, 1, \cdots, n-1$ and $j = 1, 2, \cdots, n$. That avoids r_j, where $1 \le j \le n$, to be contained in both $[(i-1)/n, i/n]$ and $[i/n, (i+1)/n]$.

Let $k = k(n)$ be the denominator of r_n, i.e. k is the unique integer such that:

$$1 + \varphi(1) + \varphi(2) + \cdots + \varphi(k-1) \le n < 1 + \varphi(1) + \varphi(2) + \cdots + \varphi(k),$$

where $\varphi(j)$ is the number of positive integers coprime to j. Then, we have

$$U(P_n, f) \le \frac{1}{n}\left(1 + \frac{\varphi(1)}{1} + \frac{\varphi(2)}{2} + \cdots + \frac{\varphi(k)}{k}\right) \le \frac{1 + \frac{\varphi(1)}{1} + \frac{\varphi(2)}{2} + \cdots + \frac{\varphi(k)}{k}}{1 + \varphi(1) + \varphi(2) + \cdots + \varphi(k-1)}.$$

From number theory, we have the following results:

$$\frac{\varphi(1)}{1} + \frac{\varphi(2)}{2} + \cdots + \frac{\varphi(k)}{k} = \frac{6k}{\pi^2} + O\left((\log k)^{2/3}(\log\log k)^{4/3}\right),$$
$$\varphi(1) + \varphi(2) + \cdots + \varphi(k-1) = \frac{3(k-1)^2}{\pi^2} + O\left((k-1)(\log(k-1))^{2/3}\log\log(k-1))^{4/3}\right).$$

Using these asymptotics, one can easily show that

$$\lim_{n\to\infty} U(P_n, f) = 0,$$

as $U(P_n, f)$ behaves like $\sim \frac{1}{k}$ as $n \to \infty$.

4.2.2 Non-Riemann integrable function: an example

The function below can be shown to be not Riemann integrable on $[0, 1]$:

$$\chi_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

To prove this, we consider an arbitrary partition P of $[0, 1]$ with partition points denoted by x_i's. As each closed interval (with positive length) contains a rational number, so we have for any i

$$\sup_{[x_{i-1}, x_i]} \chi_{\mathbb{Q}} = 1,$$

which implies

$$U(P, \chi_{\mathbb{Q}}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1.$$

However, each closed interval $[x_{i-1}, x_i]$ must also contain an irrational number, so for any i we also have

$$\inf_{[x_{i-1}, x_i]} \chi_{\mathbb{Q}} = 0,$$

and so $L(P, \chi_{\mathbb{Q}}) = 0$.

This proves

$$\mu^*(G^+(\chi_{\mathbb{Q}})) = \overline{\int_0^1} \chi_{\mathbb{Q}}(x)\,dx = \inf\{U(P, \chi_{\mathbb{Q}}) : P \text{ is a partition of } [0, 1]\} = 1$$

while

$$\mu_*(G^+(\chi_{\mathbb{Q}})) = \underline{\int_0^1} \chi_{\mathbb{Q}}(x)\,dx = \sup\{L(P, \chi_{\mathbb{Q}}) : P \text{ is a partition of } [0, 1]\} = 0.$$

Therefore, $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[0,1]$.

$\displaystyle\int_0^1 \chi_{\mathbb{Q}}(x)\,dx$ is therefore undefined in Riemann's sense. However, in MATH 3033/3043, we will introduce a more refined type of integrals, called *Lebesgue integrals*, that would allow us to integrate $\chi_{\mathbb{Q}}$ over $[0,1]$ in some other sense.

4.2.3 General bounded functions

Now we discuss the definition of Riemann integral for general bounded functions on $[a,b]$ which are not necessarily non-negative. It is not simply repeating our treatment for G^+ and applying similar rationale on G^-, because the regions G^+ and G^- must be too complicated for a general function. Consider the function

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Both G^+ and G^- have infinitely many disjoint regions (try to sketch a graph to see this!).

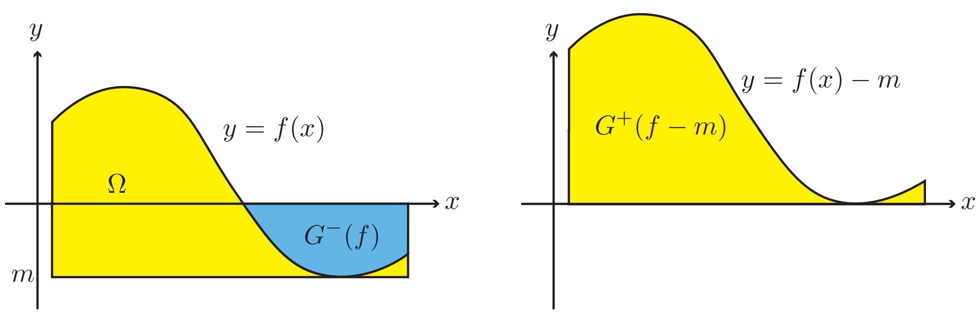
Consider $f : [a,b] \to \mathbb{R}$ which is bounded, and so one can make sense of $\inf_{x\in[a,b]} f(x) =: m$. Here we assume $m < 0$ otherwise the function f is non-negative – we have discussed that before. Note that then $f(x) - m \geq 0$ for any $x \in [a,b]$. One good observation is that

$$\mu\big(G^+(f)\big) - \mu\big(G^-(f)\big) = \mu\big(G^+(f-m)\big) + m(b-a). \tag{4.1}$$

From now on we will abbreviate $G^+_{[a,b]}$ and $G^-_{[a,b]}$ by G^+ and G^- if the interval involved is clear from the context.

Equation (4.1) can be proved by first using the translation invariance of Jordan measure, so that

$$\mu\big(G^+(f-m)\big) = \mu\,\underbrace{\{(x,y) : m \leq y \leq f(x)\}}_{G^+(f-m)+m=:\Omega}.$$



Note that $\Omega = G^+(f) \sqcup \big(([a,b]\times[m,0))\backslash G^-(f)\big)$, so we have

$$\mu(\Omega) = \mu\big(G^+(f)\big) + |m|\,(b-a) - \mu\big(G^-(f)\big),$$

and so (4.1) follows. Note that $m < 0$ in our case.

As the Riemann integral $\displaystyle\int_a^b f(x)\,dx$ is defined as $\mu\big(G^+(f)\big) - \mu\big(G^-(f)\big)$, and (4.1) relates this integral with $\int_a^b f(x) - m\,dx$ where $f(x) - m \geq 0$, we can carry over many results we proved for non-negative bounded functions to general bounded functions via (4.1).

Given any partition $P : a := x_0 < x_1 < \cdots < x_n := b$, we define just as in the non-negative case the upper and lower Darboux sums by:

$$U(P,f) := \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1}),$$
$$L(P,f) := \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1}).$$

Then, by observing that

$$\sup_I (f-m) = (\sup_I f) - m \quad \text{and} \quad \inf_I (f-m) = (\inf_I f) - m,$$

we can easily deduce that

$$\begin{aligned}
U(P,f-m) &= \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f - m \right)(x_i - x_{i-1}) = U(P,f) - m\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= U(P,f) - m(b-a), \\
L(P,f-m) &= \sum_{i=1}^{n} \left(\inf_{[x_{i-1},x_i]} f - m \right)(x_i - x_{i-1}) = L(P,f) - m\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= L(P,f) - m(b-a)
\end{aligned}$$

Using (4.1) and the above relations, one can then extend the result of Exercise 4.12 to general bounded functions:

Proposition 4.7 Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a,b]$ such that

$$\lim_{n\to\infty} U(P_n,f) = \lim_{n\to\infty} L(P_n,f) = I,$$

then f is Riemann integrable and $\displaystyle\int_a^b f(x)\,dx = I$.

Proof. Denote $m = \inf_{[a,b]} f$. Recall that

$$U(P_n,f-m) = U(P_n,f) - m(b-a) \quad \text{and} \quad L(P_n,f-m) = L(P_n,f) - m(b-a),$$

so we have

$$\lim_{n\to\infty} U(P_n,f-m) = \lim_{n\to\infty} L(P_n,f-m) = I - m(b-a).$$

Note that $f(x) - m \geq 0$ for any $x \in [a,b]$, so by Exercise 4.12 we conclude $f(x) - m$ is Riemann integrable on $[a,b]$, and so $G^+(f-m)$ is Jordan measurable and we have

$$\int_a^b (f(x) - m)\,dx = \mu\big(G^+(f-m)\big) = I - m(b-a).$$

By translational invariance, $\Omega := G^+(f-m) + m$ is also Jordan measurable with $\mu(\Omega) = I - m(b-a)$. Then, $G^+(f) = \Omega \cap ([a,b] \times [0, \sup f])$ is also Jordan measurable (here $\sup f$ means $\sup_{[a,b]} f$), and $G^-(f) = \Omega \cap ([a,b] \times [m,0])$ is also Jordan measurable. This shows, by the definition of Riemann integrals, that f is Riemann integrable on $[a,b]$. As for the value of the integral, we can use (4.1) to prove that:

$$\int_a^b f(x)\,dx = \mu\big(G^+(f)\big) - \mu\big(G^-(f)\big) = \mu\big(G^+(f-m)\big) + m(b-a) = I.$$

■

■ **Example 4.4** Show that x^3 is Riemann integrable on $[-1, \sqrt{2}]$, and find the value of the integral over $[-1, 2]$.

■ **Solution** Here we choose non-uniform partitions so that we can always make 0 as one of the partition points. For each $n \in \mathbb{N}$, we define

$$P_n : -1 < -1 + \frac{1}{n} < -1 + \frac{2}{n} < \cdots < -1 + \frac{n-1}{n} < 0 < \frac{1}{n}\sqrt{2} < \frac{2}{n}\sqrt{2} < \cdots < \frac{n-1}{n}\sqrt{2} < \sqrt{2}.$$

One can then compute that

$$\begin{aligned}
U(P_n, x^3) &= \frac{1}{n}\left(\left(-1 + \frac{1}{n}\right)^3 + \left(-1 + \frac{2}{n}\right)^3 + \cdots + \left(-1 + \frac{n-1}{n}\right)^3 + 0^3\right) \\
&\quad + \frac{1}{n}\left(\left(\frac{1}{n}\sqrt{2}\right)^3 + \left(\frac{2}{n}\sqrt{2}\right)^3 + \cdots + \left(\frac{n-1}{n}\sqrt{2}\right)^3 + \left(\frac{n}{n}\sqrt{2}\right)^3\right) \\
&= -\frac{1}{n^4}(0^3 + 1^3 + 2^3 + \cdots + (n-1)^3) + \frac{2^{3/2}}{n^4}(1^3 + 2^3 + \cdots + n^3) \\
&= -\frac{1}{n^4} \cdot \frac{(n-1)^2 n^2}{4} + \frac{2^{3/2}}{n^4} \cdot \frac{n^2(n+1)^2}{4} \to -\frac{1}{4} + \frac{2^{3/2}}{4}
\end{aligned}$$

as $n \to \infty$.

Similarly, we have

$$\begin{aligned}
L(P_n, x^3) &= -\frac{1}{n^4}(1^3 + 2^3 + \cdots + n^3) + \frac{2^{3/2}}{n^4}(0^3 + 1^3 + 2^3 + \cdots + (n-1)^3) \\
&= -\frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{2^{3/2}}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \to -\frac{1}{4} + \frac{2^{3/2}}{4}
\end{aligned}$$

as $n \to \infty$.

Using Proposition 4.7, we conclude that x^3 is Riemann integrable on $[-1, \sqrt{2}]$ and

$$\int_{-1}^{\sqrt{2}} x^3 \, dx = \frac{2^{3/2} - 1}{4}.$$

■ **Exercise 4.16** Show that for any $p \in \mathbb{N}$, the function $f(x) := x^p$ is Riemann integrable on $[a, b]$ for any real $a < b$. [Split the case into $0 \leq a < b$, $a < 0 \leq b$ and $a < b \leq 0$.]

■ **Exercise 4.17** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that the following are equivalent:

1. for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that
$$U(P, f) - L(P, f) < \varepsilon.$$
2. there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a, b]$ such that
$$\lim_{n\to\infty} (U(P_n, f) - L(P_n, f)) = 0.$$
3. there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a, b]$ so that $U(P_n, f)$ and $L(P_n, f)$ converge, and
$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f)$$
4. f is Riemann integrable on $[a, b]$ (i.e. $G_{[a,b]}(f)$ is Jordan measurable)

5. $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ (see Definition 4.4)

Suppose any one of the above (and hence all) holds, show that then any sequence of partition $\{P_n\}$ of $[a,b]$, such that both $U(P_n, f)$ and $L(P_n, f)$ converge, must satisfy:

$$\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(P_n, f).$$

■ **Exercise 4.18** Show that any monotone bounded function on $[a,b]$ must be Riemann integrable on $[a,b]$.

In view of Exercise 4.17, some textbooks would take one of (1)-(5) in that exercise to be the definition of Riemann integrability. The most common one seems to be (5).

4.2.4 Continuous functions

Using the results in Exercise 4.17, one can prove that continuous functions on $[a,b]$ must be Riemann integrable. For that we need to introduce a concept of **uniform continuity**.

To give some motivation, let's consider the function $f(x) = e^x$. It is well-known to be continuous at every $a \in \mathbb{R}$, meaning that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that whenever $|x-a| < \delta$, $|e^x - e^a| < \varepsilon$. Let's think about what δ depends on? Certainly the smaller ε is, the smaller δ is needed. Furthermore, δ also depends on a because the larger the a, the steeper the graph $y = e^x$ near a, so a smaller δ is needed. This can be seen using the mean value theorem:

$$|e^x - e^a| \le e^b\,|x-a|$$

where $b \in (a,x)$ or (x,a). If $|x-a| < \delta$, then we have

$$e^b\,|x-a| \le e^{\max\{a,x\}}\delta \le e^{\max\{a,a+\delta\}}\delta.$$

To choose δ such that $e^{\max\{a,a+\delta\}}\delta < \varepsilon$, it is impossible to make it independent of a.

Uniform continuity is a stronger notion of continuity in which the choice of δ does not depend on a specific point in the domain. Precisely, we have:

Definition 4.5 — Uniform Continuity. Let $f : I \to \mathbb{R}$ be a function defined on an interval $I = \langle a, b\rangle$. We say f is **uniformly continuous** on I if $\forall \varepsilon > 0$, there exists $\delta > 0$ which does not depend on $x, y \in I$, such that whenever $x, y \in I$ and $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

■ **Example 4.5** Any differentiable function $f : I \to \mathbb{R}$ with bounded f' on I is uniformly continuous on I. To prove this, we let $|f'(x)| \le M$ for any $x \in I$, then for any $x, y \in I$, with $x \ne y$, the mean value theorem shows there exists $\xi \in (x,y)$ or (y,x) such that

$$|f(x) - f(y)| \le |f'(\xi)|\,|x-y| \le M\,|x-y|\,.$$

$\forall \varepsilon > 0$, we choose $\delta = \frac{\varepsilon}{M+1}$, then whenever $x, y \in I$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| \le M\delta < \frac{M\varepsilon}{M+1} < \varepsilon.$$

Note that this δ does not depend on x and y. Therefore, f is uniformly continuous on I.

■ **Example 4.6** The function $f(x) = e^x$ is not uniformly continuous on $\mathbb{R}$. To see this, we assume on the contrary that it is so. Then, by taking $\varepsilon = 1$, there exists $\delta > 0$ such that whenever $|x-y| < \delta$, we have $|e^x - e^y| < 1$. Consider the sequences $x_n = n$ and $y_n = n + \frac{1}{n}$. For any $n > \frac{1}{\delta}$, we have $|x_n - y_n| = \frac{1}{n} < \delta$, and so $\left|e^n - e^{n+\frac{1}{n}}\right| < 1$. By mean value theorem, there exists $z_n \in (x_n, y_n)$ such that

$$\left|e^n - e^{n+\frac{1}{n}}\right| = e^{z_n} \cdot \frac{1}{n} \geq \frac{e^n}{n}.$$

However, this would show

$$\frac{e^n}{n} < 1$$

for any $n > \frac{1}{\delta}$. It is a contradiction as $\frac{e^n}{n} \to +\infty$ as $n \to \infty$. Therefore, e^x is not uniformly continuous on $\mathbb{R}$.

However, e^x is uniformly continuous on any bounded interval by the previous example, as it has bounded derivative on any bounded interval.

This above example of e^x shows that whether a function is uniform continuous depends on the domain. A function can be uniformly continuous on a smaller domain but not on a larger one. Therefore, it is crucial the specify the domain, such as f is uniformly continuous on $(a, b]$, when we mention about uniform continuity.

■ **Exercise 4.19** Show that x^2 is uniformly continuous on any bounded interval, but not on $\mathbb{R}$.

One important fact relating Riemann integrals of continuous functions is that continuous functions on any **closed** and **bounded** interval must be uniformly continuous on that interval.

Proposition 4.8 Any continuous function $f : [a, b] \to \mathbb{R}$ on a closed and bounded interval $[a, b]$ must be uniformly continuous on $[a, b]$.

Proof. The proof is to use Bolzano-Weierstrass's Theorem. Assume it is not true that f is uniformly continuous on $[a, b]$, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, there exists $x_\delta, y_\delta \in [a, b]$ with $|x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$.

In particular, for any $n \in \mathbb{N}$, there exists $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

As $[a, b]$ is closed and bounded, there exist convergent subsequences $\{x_{n_k}\}_{k=1}^\infty$ and $\{y_{n_k}\}_{k=1}^\infty$. Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ for any k, we have $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}$. Denote the limit by L, and by closedness of $[a, b]$ we have $L \in [a, b]$ too.

Recall that f is continuous on $[a, b]$, and in particular at L, so we have

$$\lim_{k\to\infty} f(x_{n_k}) = f(L) \quad \text{and} \quad \lim_{k\to\infty} f(y_{n_k}) = f(L).$$

However, that would imply

$$0 < \varepsilon_0 \leq \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = |f(L) - f(L)| = 0$$

which is clearly absurd.

This proves f must be uniformly continuous on $[a, b]$. ■

Proposition 4.9 Any continuous function f on a closed and bounded interval $[a, b]$ must be Riemann integrable on $[a, b]$.

Proof. By Proposition 4.8, f is uniformly continuous on $[a, b]$. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in [a, b]$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Now, take a partition P, with partition points $\{x_i\}_{i=0}^n$ of $[a, b]$ such that each subdivision $[x_{i-1}, x_i]$ has length $< \delta$, then we have for any $x, y \in [x_{i-1}, x_i]$, $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$. This shows

$$\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \leq \frac{\varepsilon}{2(b-a)}.$$

Then, we have

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right)(x_i - x_{i-1}) < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon.$$

By Exercise 4.17, f is Riemann integrable on $[a,b]$. ■

4.2.5 Properties of Riemann integrals

There are several properties about Riemann integrals that we will frequently use:

Proposition 4.10 — Properties of Riemann Integrals. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be bounded functions. Then,

1. Fix any $c \in (a,b)$. If f is Riemann integrable on $[a,c]$ and on $[c,b]$, then f is Riemann integrable on $[a,b]$ and
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$
2. If f is Riemann integrable on $[a,b]$, then so does cf for any $c \in \mathbb{R}$ and
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$
3. If f and g are both Riemann integrable on $[a,b]$, then so does $f+g$ and we have
$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$
4. If f and g are Riemann integrable on $[a,b]$ and $f(x) \le g(x)$ for any $x \in [a,b]$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$
5. If f is Riemann integrable on $[a,b]$, then so does $|f|$ and
$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Proof. (1) follows from the fact that $G^{\pm}_{[a,b]}(f) = G^{\pm}_{[a,c]}(f) \cup G^{\pm}_{[c,b]}(f)$, and so if $G^{\pm}_{[a,c]}(f)$ and $G^{\pm}_{[c,b]}(f)$ are Jordan measurable, then the union $G^{\pm}_{[a,b]}(f)$ is also Jordan measurable. The intersection $G^{\pm}_{[a,c]}(f) \cap G^{\pm}_{[c,b]}(f)$ is a line segment, so it has Jordan measure zero. This proves

$$\mu(G^{\pm}_{[a,b]}(f)) = \mu(G^{\pm}_{[a,c]}(f) \cup G^{\pm}_{[c,b]}(f)) = \mu(G^{\pm}_{[a,c]}(f)) + \mu(G^{\pm}_{[c,b]}(f)),$$

which implies

$$\begin{aligned}\int_a^b f(x)\,dx &= \mu(G^{+}_{[a,b]}(f)) - \mu(G^{-}_{[a,b]}(f)) \\ &= \mu(G^{+}_{[a,c]}(f)) + \mu(G^{+}_{[c,b]}(f)) - \mu(G^{-}_{[a,c]}(f)) - \mu(G^{-}_{[c,b]}(f)) \\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.\end{aligned}$$

(2) follows directly from

$$U(P,cf) = \begin{cases} cU(P,f) & \text{if } c \ge 0 \\ cL(P,f) & \text{if } c < 0 \end{cases} \text{ and } L(P,cf) = \begin{cases} cL(P,f) & \text{if } c \ge 0 \\ cU(P,f) & \text{if } c < 0 \end{cases},$$

and the results proved in Exercise 4.17.

To prove (3), we make sure of the fact that

$$\sup_I (f+g) \leq \sup_I f + \sup_I g, \quad \text{and} \quad \inf_I (f+g) \geq \inf_I f + \inf_I g.$$

[The proof of these is trivial: $\sup_I f + \sup_I g$ is an upper bound of $f+g$, and $\inf_I f + \inf_I g$ is a lower bound of $f+g$.]

This shows for any partition P of $[a,b]$, we have:

$$\begin{aligned} U(P, f+g) &\leq U(P,f) + U(P,g), \\ L(P, f+g) &\geq L(P,f) + L(P,g). \end{aligned}$$

Given that both f and g are Riemann integrable on $[a,b]$, by Exercise 4.17, there exist sequences of partitions $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ of $[a,b]$ such that

$$\lim_{n\to\infty} \big(U(P_n,f) - L(P_n,f)\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} \big(U(Q_n,g) - L(Q_n,g)\big).$$

Consider the sequence of partition $R_n := P_n \cup Q_n$ (i.e. mixing the partition points of P_n and Q_n are create a more refined partition), one can show that $U(R_n,f) - L(R_n,f) \leq U(P_n,f) - L(P_n,f)$ and $U(R_n,g) - L(R_n,g) \leq U(Q_n,g) - L(Q_n,g)$. See Exercise 4.20.

Combining with previous results, we get

$$\begin{aligned} U(R_n, f+g) - L(R_n, f+g) &\leq U(R_n,f) + U(R_n,g) - L(R_n,f) - L(R_n,g) \\ &\leq \underbrace{U(P_n,f) - L(P_n,f)}_{\to 0} + \underbrace{U(Q_n,g) - L(Q_n,g)}_{\to 0} \end{aligned}$$

as $n \to \infty$. By Exercise 4.17, $f+g$ is Riemann integrable on $[a,b]$. To prove the additivity of integrals, we take a subsequence $\{R_{n_k}\}_{k=1}^{\infty}$ of $\{R_n\}_{n=1}^{\infty}$ such that all of the following converge as $k \to \infty$:

$$U(R_{n_k}, f), L(R_{n_k}, f), U(R_{n_k}, g), L(R_{n_k}, g), U(R_{n_k}, f+g), L(R_{n_k}, f+g).$$

It is possible by Bolzano-Weierstrass's Theorem (note that f and g are bounded). According to Exercise 4.17, we have

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} U(R_{n_k}, f) = \lim_{k\to\infty} L(R_{n_k}, f)$$

and the same for g.

Then by Exercise 4.20 below, we have for any k:

$$\begin{aligned} &L(R_{n_k}, f) + L(R_{n_k}, g) \leq L(R_{n_k}, f+g) \\ &\leq \underline{\int_a^b} (f(x)+g(x))\,dx \leq \overline{\int_a^b} (f(x)+g(x))\,dx \\ &\leq U(R_{n_k}, f+g) \leq U(R_{n_k}, f) + U(R_{n_k}, g). \end{aligned}$$

Letting $k \to \infty$, we get

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \leq \underline{\int_a^b} (f(x)+g(x))\,dx \leq \overline{\int_a^b} (f(x)+g(x))\,dx \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

This proves (3) completely.

(4) is obvious by the fact that if $f(x) \leq g(x)$ for any $x \in [a,b]$, then $\sup_I f \leq \sup_I g$ for any interval $I \subset [a,b]$ and so $U(P,f) \leq U(P,g)$ for any partition P of $[a,b]$.

For (5), note that $G(|f|) = G^+(f) \cup R_x\big(G^-(f)\big)$ where $R_x : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the x-axis, which is an isometry. If f is Riemann integrable, then $G^+(f)$ and $G^-(f)$ are

Jordan measurable, so $R_x(G^-(f))$ and hence $G(|f|)$ is also Jordan measurable. This shows $|f|$ is Riemann integrable on $[a,b]$. The inequality

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\;dx$$

follows directly from $-|f(x)| \leq f(x) \leq |f(x)|$ and the use of (4). ■

Combining (2) and (3) of Proposition 4.10, i.e. by taking $c = -1$ in (2), one can also prove that if f and g are both Riemann integrable on $[a,b]$, then so does $f-g$ and we have

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

When $a > b$, we would define

$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx.$$

Using this definition, one can easily show that (1) of Proposition 4.10 also holds if even c is not in (a,b).

■ **Example 4.7** Suppose f is continuous on $[a,b]$, hence Riemann integrable on $[a,b]$. Show that there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

■ **Solution** By extreme value theorem, f achieves its maximum and minimum on $[a,b]$. Let

$$M := \sup_{[a,b]} f = f(x_1) \quad \text{and} \quad m := \inf_{[a,b]} f = f(x_2)$$

for some $x_1, x_2 \in [a,b]$.

Then by $f(x) \leq M$ for any $x \in [a,b]$, (4) in Proposition 4.10 shows

$$\int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a).$$

Similarly by $f(x) \geq m$ for any $x \in [a,b]$, we have

$$m(b-a) \leq \int_a^b m\,dx \leq \int_a^b f(x)\,dx.$$

This shows

$$f(x_2) = m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M = f(x_1).$$

As f is continuous, intermediate value theorem shows there exists c between x_1 and x_2 such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

■ **Exercise 4.20** Show that for any bounded function $f : [a,b] \to \mathbb{R}$ and any partition P of $[a,b]$, if we add one partition point c to P, and denote $P' = P \cup \{c\}$, then

$$L(P, f) \leq L(P', f) \leq U(P', f) \leq U(P, f).$$

■ **Exercise 4.21** Let I be a closed and bounded interval, and $J \subset I$ be another closed and bounded interval. Show that if f is Riemann integrable on I, then it is also Riemann integrable on J.

Assume further that $f(x) \geq 0$ on $[a, b]$, show that

$$\int_{\alpha}^{\beta} f(x)\,dx \leq \int_{\gamma}^{\eta} f(x)\,dx$$

if $[\alpha, \beta] \subset [\gamma, \eta] \subset [a, b]$.

■ **Exercise 4.22** Show that if $|f(x)| \leq M$ for any $x \in [a, b]$, then

$$\left|f(x)^2 - f(y)^2\right| \leq 2M\,|x - y| \quad \forall x, y \in [a, b].$$

Hence, show that if f is Riemann integrable on $[a, b]$, then so does f^2.

Using this and the properties of Riemann integrabs proven, show that if f, g are bounded Riemann integrable functions on $[a, b]$, then so does fg.