香港科技大學 THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

數學系 DEPARTMENT OF MATHEMATICS

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# FINAL EXAMINATION

**Course Code:** MATH 4051
**Course Title:** Theory of Ordinary Differential Equations
**Semester:** Spring 2019-20
**Date and Time:** 1:30-4:30PM, 2 June 2020

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## Instructions

- It is an **OPEN-NOTES** exam. You can look at any materials, both online and offline. However, only results discussed in class, or proved in homework can be directly quoted.
- Discussion with any person (online or offline) is **strictly prohibited**, and is a serious violation of the honor code. Posting related questions in any online forum is also a serious violation of the honor code.
- Answer **ALL** problems. Write your solutions in your own paper. Submit the file as a PDF to Canvas by 4:50PM today.
- You must **SHOW YOUR WORK**, **JUSTIFY YOUR ARGUMENTS**, and **PRESENT CLEARLY** in order to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

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## HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

**Copy the following statement and sign on the first page of your answer sheets:**

"I, YOUR FULL NAME (HKUST ID), confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

YOUR SIGNATURE

## Part A - Short Questions (25 points)

[Recommended timing: $< 30$ minutes]

1. Suppose $A$ is a $2 \times 2$ **real** matrix such that: **[6]**

$$A \begin{bmatrix} 1+i \\ 3-2i \end{bmatrix} = (-1+4i) \begin{bmatrix} 1+i \\ 3-2i \end{bmatrix}.$$

Answer the following questions. No justification is needed.

(a) Write down $e^A$ as a finite product of explicit real matrices. $= \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-1}\cos 4 & e^{-1}\sin 4 \\ -e^{-1}\sin 4 & e^{-1}\cos 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}^{-1}$

(b) Consider the system $\mathbf{x}' = A\mathbf{x}$. Is the origin stable, asymptotically stable, or unstable?

(c) What is $\omega(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \mathbb{R}^2$? $\omega(x_0) = \{0\}$ $\forall x_0 \in \mathbb{R}^2$.

2. The following are **unordered** steps in the proof of the Picard-Lindelöf's Existence Theorem. Denote $\mathbf{x}_n(t)$ to be the Picard's iteration sequence associated to the IVP $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$ where $\mathbf{F}$ is locally Lipschitz continuous on $\mathbb{R}^d$. **[4]**

Arrange the whole proof in the correct logical order.

1. Show that $\mathbf{F}(\mathbf{x}_n)$ converges uniformly on $[-\varepsilon, \varepsilon]$ to the limit $\mathbf{F}(\mathbf{x}_\infty)$.
2. Show that $\sum_{n=1}^{\infty}(\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t))$ converges uniformly on $t \in [-\varepsilon, \varepsilon]$.
3. Show that there exists $\epsilon > 0$ such that $\mathbf{x}_n(t) \in B_r(\mathbf{x}_0)$ for any $n \geq 0$ and $t \in [-\varepsilon, \varepsilon]$.
4. Show that there exists $r > 0$ such that $\mathbf{F}$ is Lipschitz continuous on $B_r(\mathbf{x}_0)$.
5. Show by induction that $|\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)| \leq \dfrac{KL^{n-1}|t|^n}{n!}$ for any $n \geq 1$ and $t \in [-\varepsilon, \varepsilon]$ where $K$ and $L$ are some positive constants.
6. Show that $\mathbf{x}_n(t)$ converges uniformly on $t \in [-\varepsilon, \varepsilon]$ to a limit function $\mathbf{x}_\infty(t)$.
7. Show that $\mathbf{x}_\infty(t)$ is a continuous solution to the integral equation associated to the IVP, and hence is a solution to the IVP.

Logical order in the proof: 4 $\rightarrow$ 3 $\rightarrow$ 5 $\rightarrow$ 2 $\rightarrow$ 6 $\rightarrow$ 1 $\rightarrow$ 7.

[Grading: all correct +4, minor mistakes +3, otherwise 0]

3. Consider a system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ where $\mathbf{F}$ is $C^1$ on $\mathbb{R}^d$. Discuss whether each statement below is true. If true, explain why briefly. If false, give a counter-example.

(a) If there exists $M > 0$ such that such that $|\mathbf{F}(\mathbf{x})| \leq M$ for any $\mathbf{x} \in \mathbb{R}^d$, then any solution to the system must be defined on $t \in [0, \infty)$. **[3]**

(b) If $\sup_{\mathbf{x} \in \mathbb{R}^d} |\mathbf{F}(\mathbf{x})| = +\infty$, then any solution to the system must encounter finite-time singularity. **[3]**

(a) True $|x(t) - x(0)| = \left| \int_0^t x' \right| = \left| \int_0^t F \right| \leq \int_0^t \underbrace{|F|}_{M} \leq Mt \quad \forall t > 0$.

$\Rightarrow |x(t)| \leq |x(0)| + Mt$

$\therefore$ On any bounded time interval $[0,T]$, $|x(t)| \leq |x(0)| + MT$ $\forall t \in [0,T]$.

$\rightsquigarrow$ bounded solution cannot have finite-time singularity.

$\therefore$ $x(t)$ can be extended beyond any bounded time interval $[0,T]$ $\Rightarrow$ $x(t)$ is defined on $t \in [0,\infty)$.

(b) False. Counter-example $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -y \\ x \end{bmatrix} \leftarrow F(x,y)$. Clearly $|F(x,y)| = \sqrt{x^2+y^2} \Rightarrow \sup_{(x,y)\in\mathbb{R}^2} |F(x,y)| = +\infty$. But all solutions are defined on $(-\infty,\infty)$ as it is linear.

4. Let $\Omega$ be an open subset of $\mathbb{R}^d$, and assume $\Omega \neq \mathbb{R}^d$. Let $\mathbf{F}(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^d$ be a time-independent vector field and $\mathbf{x}_0 \in \Omega$. Consider the (IVP): $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$.

   Complete the diagram below according to the following instructions:

   i. For each straight-line in the diagram, draw an arrow ($\to$ or $\leftarrow$) to indicate which box implies which box, i.e. "A $\to$ B" means: "A implies B". **[5]**

   ii. Using an arrow, connect "short-time existence for (IVP)" to **exactly one** box below it. Choose the best box to connect. **[2]**

   iii. Using an arrow, connect "long-time existence for (IVP)" to **exactly one** box above it. Again, choose the best box to connect. **[2]**

$$
\begin{tikzcd}
 & \text{Short-time existence for (IVP)} & \\
\mathbf{F} \text{ is } C^1 \text{ on } \Omega \arrow[r] & \mathbf{F} \text{ is locally Lipschitz on } \Omega \arrow[u] & \mathbf{F} \text{ is Lipschitz on } \Omega \arrow[l] \\
\mathbf{F} \text{ is } C^1 \text{ on } \mathbb{R}^d \arrow[u] \arrow[r] & \mathbf{F} \text{ is locally Lipschitz on } \mathbb{R}^d \arrow[u] & \mathbf{F} \text{ is Lipschitz on } \mathbb{R}^d \arrow[l] \arrow[u] \arrow[dl] \\
 & \text{Long-time existence for (IVP)} &
\end{tikzcd}
$$

## Part B - Long Questions (75 points): Answer ALL problems

[Recommended timing: Q1 $< 30$m, Q2 $< 30$m, Q3 $< 45$m, Q4 $< 45$m]

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1. Let $A$ be a constant $d \times d$ real matrix, and $\mathbf{b}, \mathbf{x}_0 \in \mathbb{R}^d$ are constant vectors. Consider the initial-value problem in $\mathbb{R}^d$:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

   Let $\{\mathbf{x}_n(t)\}_{n=0}^{\infty}$ be the Picard's iteration sequence associated to this IVP, defined as in the lecture notes.

   (a) By computing the first few terms $\mathbf{x}_n(t)$'s, guess the general term and justify it by induction. **[15]**

   (b) Assume that $A$ is invertible. Using (a), find the explicit solution to the given IVP. You can express your answer in terms of matrix exponentials. Verify that it is indeed a solution to the given IVP. **[4]**

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2. Regard $\mathbb{R}^{m+n} = \mathbb{R}^m \oplus \mathbb{R}^n$ and write its element as $\mathbf{x} = (x, y)$ where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Denote the projection maps by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Consider the system in $\mathbb{R}^{m+n}$: **[12]**

$$\mathbf{x}' = A\mathbf{x} + \mathbf{h}(\mathbf{x})$$

where $A$ is a real $(m+n) \times (m+n)$ matrix of the form:

$$A = \begin{bmatrix} N & 0 \\ 0 & P \end{bmatrix}.$$

Here $N$ is an $m \times m$ real matrix with all eigenvalues negative, and $P$ is an $n \times n$ real matrix with all eigenvalues positive.

Fix $p \in \mathbb{R}^m$. Show that for a certain choice of $q \in \mathbb{R}^n$, the initial-value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{h}(\mathbf{x}), \quad \mathbf{x}(0) = (p, q) \text{ where } |\mathbf{h}(\mathbf{x})| = o(|\mathbf{x}|) \text{ as } \mathbf{x} \to 0,$$

is equivalent to the integral equation

$$\mathbf{x}(t) = e^{tA} \begin{bmatrix} p \\ 0 \end{bmatrix} + \int_0^t e^{(t-s)A} \pi_1(\mathbf{h}(\mathbf{x}(s)))\, ds - \int_t^\infty e^{(t-s)A} \pi_2(\mathbf{h}(\mathbf{x}(s)))\, ds.$$

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3. Let $p(x)$ be an odd degree polynomial such that the highest order term has positive coefficient. Consider the system in $\mathbb{R}^2$:

$$\begin{aligned} x' &= y \\ y' &= -y^3 - p(x) \end{aligned}$$

   (a) Find an even degree polynomial $f(x)$ such that: **[6]**

$$\frac{d}{dt}(f(x(t)) + y(t)^2) \leq 0$$

   for any solution $(x(t), y(t))$ to the system.

   (b) Hence, show that any solution $(x(t), y(t))$ to the system is bounded on $t \in [0, \infty)$. **[6]**

   (c) Suppose $p$ has a unique real root $x_*$. Show that for any $\varepsilon > 0$, there exists $(x_0, y_0) \in B_\varepsilon\big((x_*, 0)\big)$ such that for any $t > 0$, there exists $\tau > t$ such that $\varphi_\tau(x_0, y_0) \in B_\varepsilon\big((x_*, 0)\big)$. Here $\varphi_t$ denotes the flow map of the system. **[12]**
   [Hint: Proof by contradiction, and use (b).]

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4. Let $A$ be a $2 \times 2$ real symmetric matrix whose eigenvalues are negative. From linear algebra, we know that there exists an orthogonal matrix $P$ (i.e. $P^T P = I$) and a diagonal matrix $D = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$ such that $A = PDP^T$. For simplicity, we denote $Q = -A^{-1}$.

   Consider the system in $\mathbb{R}^2$:
$$\mathbf{x}' = A\mathbf{x} + \mathbf{h}(\mathbf{x})$$
   where $\mathbf{h}(\mathbf{x})$ is $C^1$ on $\mathbb{R}^2$ satisfying $|\mathbf{h}(\mathbf{x})| = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to 0$ and $\mathbf{h}(\mathbf{0}) = \mathbf{0}$.

   (a) Show that $\mathbf{h}(\mathbf{x}) \cdot Q\mathbf{x} = o(|\mathbf{x}|^2)$ and $\mathbf{x} \cdot Q\mathbf{h}(\mathbf{x}) = o(|\mathbf{x}|^2)$ as $|\mathbf{x}| \to 0$. **[6]**

   (b) Let $L : \mathbb{R}^2 \to \mathbb{R}$ be defined as: **[12]**
$$L(\mathbf{x}) := \mathbf{x} \cdot Q\mathbf{x}.$$
   Show that $L$ (when restricted on a small open ball around $\mathbf{0}$) is a strict Lyapunov function for $\mathbf{0}$.

   (c) Results from (b) give another proof of the Poincaré-Lyapunov's Theorem for the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ in the special case $D\mathbf{F}_{\mathbf{x}_*}$ being symmetric. Explain why the proof would fail if one of the eigenvalues of $D\mathbf{F}_{\mathbf{x}_*}$ is zero. **[2]**

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* End of Paper *

** **

{Part B} Sketch only

**PROBLEM #1**

(a) $$\vec{x}_n(t) = \sum_{k=0}^{n} \frac{t^k A^k}{k!}\vec{x}_0 + \sum_{k=1}^{n} \frac{t^k A^{k-1}}{k!}\vec{b}$$

(b) The solution is given by

$$\begin{aligned}\vec{x}_\infty(t) &:= \lim_{n\to\infty} \vec{x}_n(t)\\ &= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{(tA)^k}{k!}\vec{x}_0 + A^{-1}\left(\sum_{k=0}^{n}\frac{(tA)^k}{k!} - \underset{\substack{\uparrow\\ k=0}}{I}\right)\vec{b}\right)\\ &= e^{tA}\vec{x}_0 + A^{-1}(e^{tA}-I)\vec{b}.\end{aligned}$$

**PROBLEM #2**

The integral equation can be written as (note that $e^{tA} = \begin{bmatrix} e^{tN} & 0 \\ 0 & e^{tP}\end{bmatrix}$)

$$\vec{x}(t) = e^{tA}\begin{bmatrix} p \\ 0 \end{bmatrix} + \int_0^t e^{(t-s)A}\pi_1(h(x(s)))\,ds$$

$$\left.\begin{aligned} &+ \int_0^t e^{(t-s)A}\pi_2(h(x(s)))\,ds \\ &- \int_0^\infty e^{(t-s)A}\underbrace{\pi_2(h(x(s)))}_{\text{of the form } \begin{bmatrix} 0 \\ * \end{bmatrix}}\,ds \end{aligned}\right\} = -\int_t^\infty$$

$$\Leftrightarrow \quad \vec{x}(t) = \begin{bmatrix} e^{tN}p \\ 0 \end{bmatrix} + \int_0^t e^{(t-s)A} h(x(s))\,ds + \begin{bmatrix} 0 \\ -e^{tP}\int_0^\infty e^{-sA}\pi_2(h(x(s)))\,ds \end{bmatrix}$$

$$\Leftrightarrow \quad \vec{x}(t) = \underbrace{\begin{bmatrix} e^{tN} & \\ 0 & e^{tP} \end{bmatrix}}_{=e^{tA}}\begin{bmatrix} p \\ -\int_0^\infty e^{-sA}\pi_2(h(x(s)))\,ds \end{bmatrix} + \int_0^t e^{(t-s)A} h(x(s))\,ds. \qquad \text{(IE)}$$

We let $q := -\int_0^\infty e^{-sA}\pi_2(h(x(s)))\,ds$, then we claim that the (IVP) $\vec{x}' = Ax + h$, $\vec{x}(0) = \begin{bmatrix} p \\ q \end{bmatrix}$ is equivalent to (IE).

(IVP) $\Rightarrow$ (IE): consider $(e^{-tA}x)' = e^{-tA}h$, then

$$\text{RHS} = e^{tA}\begin{bmatrix} p \\ -\int_0^\infty e^{-sA}\pi_2(h(x(s)))\,ds \end{bmatrix} + e^{tA}\int_0^t (e^{-sA}x)'(s)\,ds$$

$$= e^{tA}\Big[\underbrace{-\int_0^\infty e^{-sA}\pi_2(h(x(s)))\,ds}_{\cancel{x(0)}}\Big]^{P} + e^{tA}\Big[e^{-tA}x(t) - \cancel{x(0)}\Big]$$

$$= x(t)$$
$$= \text{LHS}.$$

(IE) $\Rightarrow$ (IVP) : Fundamental Theorem of Calculus.

PROBLEM #3

(a) $$\frac{d}{dt}\left(f(x(t)) + y(t)^2\right) = f'(x(t))\,x'(t) + 2y(t)\,y'(t)$$
$$= f'(x(t))\,y(t) - 2y(t)\left(y^3(t) + p(x(t))\right)$$
$$= y(t)\left(f'(x(t)) - 2p(x(t))\right) \underbrace{-2y(t)^4}_{\leq 0}.$$

Pick $f(x) = \int_0^x 2p(z)\,dz$, then $f'(x) = 2p(x)$.

$$\Rightarrow \frac{d}{dt}\left(f(x(t)) + y(t)^2\right) = -2y(t)^4 \leq 0$$

$p$ = odd degree polynomial $\Rightarrow$ $\int p$ is an even degree polynomial.

(b) let $f(x) = \underbrace{a_{2n}}\, x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$

$\uparrow$ $a_{2n} > 0$ as leading coefficient of $p(x)$ is positive.

then $$\lim_{x\to\pm\infty} \frac{f(x)}{x^{2n}} = a_{2n}$$

$$\Rightarrow \exists R > 0 \text{ s.t. } |x| > R \Rightarrow \frac{1}{2}a_{2n} < \frac{f(x)}{x^{2n}} < 2a_{2n}.$$

For any solution $(x(t), y(t))$, from (a) we have

$$f(x(t)) + y(t)^2 \leq f(x(0)) + y(0)^2 =: C \qquad \forall t \geq 0.$$

- If $|x(t)| > R$, then $\frac{1}{2}x(t)^{2n}a_{2n} < f(x(t)) \leq f(x(t)) + y(t)^2 \leq C.$

$$\Rightarrow \underline{|x(t)| < \sqrt[n]{\frac{2C}{a_{2n}}} =: \tilde{C}}.$$

$$y(t)^2 \leq \frac{1}{2}a_{2n}x^{2n} + y^2 \leq f(x(t)) + y(t)^2 \leq C$$
$$\Rightarrow \underline{|y(t)| \leq C}$$

- If $|x(t)| \leq R$, then $f(x(t)) + y(t)^2 \leq C$

$$\Rightarrow y(t)^2 \leq C - \underbrace{f(x(t))}_{\text{bounded as } f \text{ is continuous.}} \leq C'$$

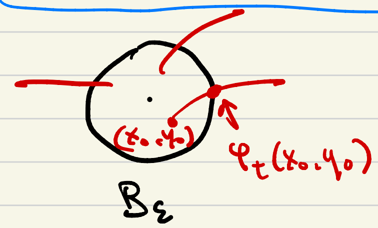
$\therefore$ In any case, $x(t), y(t)$ are bounded.

(c) Equilibrium points of the system are of form: $(x_*, 0)$ where $p(x_*) = 0$.

$\therefore$ By the given condition, the system has only one equilibrium.

Suppose otherwise that $\exists \varepsilon > 0$, $\forall (x_0, y_0) \in B_\varepsilon((x_*, 0))$ s.t. $\exists t > 0$ and $\forall \tau > t$, we have $\varphi_\tau(x_0, y_0) \notin B_\varepsilon((x_*, 0))$.

($\dagger$)




Pick any solution $(x(t), y(t)) \neq (x_*, 0)$, by (b) it is bounded.

let $|x(t)|, |y(t)| \leq C$, then combine with the above:

$K := \{(x,y) \mid |x| \leq C, |y| \leq C\} - B_\varepsilon((x_*, 0))$ traps the trajectory $(x(t), y(t))$, because if it ever enters $B_\varepsilon((x_*, 0))$, it would leave forever after some finite-time.

$K$ is clearly compact, without equilibrium point

$\xRightarrow{\text{Poincaré-Bendixson}}$ $\exists$ non-trivial periodic solution in $K$. — (*)

However, $\operatorname{div}\begin{bmatrix} y \\ -y^3 - p(x) \end{bmatrix} = \frac{\partial y}{\partial x} + \frac{\partial}{\partial y}(-y^3 - p(x)) = -3y^2 \leq 0$. (and $= 0$ a.e.) in $\mathbb{R}^2$.

Bendixson-Dulac $\Rightarrow$ no non-trivial periodic solution in $\mathbb{R}^2$! — (**)

(*) and (**) contradict each other.

$\therefore$ ($\dagger$) is false. $\boxtimes$

## PROBLEM #4

(a) $$\frac{|h(x)\cdot Qx|}{|x|^2} \le \frac{|h(x)|\,|Qx|}{|x|^2} \le \underbrace{\frac{|h(x)|}{|x|}}_{\to 0}\cdot\|Q\|\frac{\cancel{|x|}}{\cancel{|x|}} \to 0 \quad \text{as } |x|\to 0$$

$$\frac{|x\cdot Qh(x)|}{|x|^2} = \frac{|Qh(x)|}{|x|} \le \|Q\|\frac{|h(x)|}{|x|} \to 0 \quad \text{as } |x|\to 0.$$

$$\frac{|h(x)|}{|x|} \to 0$$

(b) Clearly $L(0)=0$,

eigenvalues of $Q$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} > 0$.

$\therefore$ $L(\vec{x}) = x\cdot Qx \ge 0$ (and equality holds $\Leftrightarrow$ $x=0$).

$$\frac{d}{dt}L(\underbrace{\vec{x}(t)}_{\text{solution}}) = x'\cdot Qx + x\cdot Qx' = (Ax+h)\cdot Qx + x\cdot Q(Ax+h)$$

$$= Ax\cdot Qx + \underbrace{h\cdot Qx}_{=o(|x|^2)} + x\cdot QAx + \underbrace{x\cdot Qh}_{=o(|x|^2)}$$

$$= PDP^Tx\cdot(-PD^{-1}P^T)x + x\cdot(-A^{-1}Ax) + o(|x|^2)$$

$$= (PDP^Tx)^T(-PD^{-1}P^T)x - |x|^2 + o(|x|^2)$$

$$= -x^TP\,D\underbrace{P^TP}_{=I}D^{-1}P^Tx - |x|^2 + o(|x|^2)$$

$$x^TP\,DP^TPD^{-1}P^Tx = x^Tx = |x|^2$$

$$= -2|x|^2 + o(|x|^2).$$

$\exists\,\delta>0$ s.t. the $o(|x|^2)$ term $\le |x|^2$ whenever $x\in B_\delta(0)$.

$\therefore$ $\frac{d}{dt}L(x(t)) \le -|x|^2 \le 0$ whenever $x\in B_\delta(0)$.
(equation holds $\Leftrightarrow$ $x(t)=0$)

$\therefore$ $L$ is a strictly Lyapunov function when restricted to $B_\delta(0)$.

(c) If $DF_{x_*}$ has one eigenvalue $=0$, then the linearization $\vec{x}' = DF_{x_*}\cdot\vec{x} + h(\vec{x})$ would have $A = DF_{x_*}$ being not invertible. $Q$ cannot be defined.