- MATH 4051 : Tutorial 12 (18 - 05 - 2020)
  - Brief discussion on Lorenz System, attractor
  - General Review and Q&A. (If time allowed)

- Lorenz System and Attractor
  - What's Lorenz System?

A simplified model for atmosphere convection, developed by Edward Lorenz in 1963.

$$\begin{cases} x' = \sigma(y-x) & (1) \\ y' = \rho x - y - xz & (2) \\ z' = xy - \beta z & (3) \end{cases}$$

Commonly-used Parameter
$\sigma = 10$, $\rho = 28$, $\beta = 8/3$.

Chaotic Behavior

where $\sigma, \rho, \beta > 0$.

OR we may write it as $\vec{x}' = \mathcal{L}(\vec{x})$

- Equilibrium points : Solve $\mathcal{L}(\vec{x}) = \vec{0}$.

From (1), we have $y = x$,

then (2) $\Rightarrow$ $x(\rho - 1 - z) = 0$

If $x = 0 \Rightarrow y = 0 \Rightarrow z = 0$ (by (3)).

If $x \neq 0$, then $z = \rho - 1$.

(3) $\Rightarrow$ $x^2 = \beta z = \beta(\rho - 1)$

$\Rightarrow x = y = \pm\sqrt{\beta(\rho-1)}$ (Need $\rho > 1$.)

Three equilibrium points : (Given that $\rho > 1$)

$\vec{x}_e = (0, 0, 0)$;

$\vec{x}_+ = \left(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1\right)$

$\vec{x}_- = \left(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1\right)$

- If $\rho \leq 1$, only one equilibrium point : $\vec{x}_e$.

- Stability of equilibrium points

$$DL = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

- $\vec{x_e} = (0, 0, 0)$

$$DL_{\vec{x}_e} = \left[\begin{array}{cc:c} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ \hdashline 0 & 0 & -\beta \end{array}\right]$$

$$\boxed{\begin{aligned} &\begin{vmatrix} \lambda+\sigma & -\sigma \\ -\rho & \lambda+1 \end{vmatrix} \\ &= (\lambda+\sigma)(\lambda+1) - \sigma\rho \\ &= \lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho) \end{aligned}}$$

$\Rightarrow \quad \lambda_1 = -\beta ,$

$$\lambda_+ = \frac{1}{2}\left(-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}\right),$$

$$\lambda_- = \frac{1}{2}\left(-(\sigma+1) - \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}\right)$$

① If $0 \le \rho < 1$ (only one equilibrium case), then

$(\sigma+1)^2 - 4\sigma(1-\rho) < (\sigma+1)^2$

$\Rightarrow \quad \lambda_+ < 0 , \quad \lambda_- < -(1+\sigma) < 0$

- All eigenvalues are negative, by Poincare-Lyapunov's theorem, $(0, 0, 0)$ is asymtotically stable.

$$\boxed{\begin{aligned} &\underline{\text{OR}} \text{ consider } L : \mathbb{R}^3 \to \mathbb{R}, \text{ given by } L(x, y, z) = x^2 + \sigma y^2 + \sigma z^2. \\ &\text{which shows } \vec{x_e} \text{ is a Global Attractor for } \rho \in [0, 1). \end{aligned}}$$

② If $\rho = 1$, the eigenvalues of $DL_{(0,0,0)}$ are $-\beta$, $\underline{0}$, $-(1+\sigma)$.

Difficult to analysis. Omitted.

But at this case, $\vec{x}_+$, $\vec{x}_-$ appears and coincide with $\vec{x_e}$.

③ If $\rho > 1$, $(\sigma+1)^2 - 4\sigma(1-\rho) > (\sigma+1)^2$

$$\Rightarrow \quad \lambda_+ = \frac{1}{2}\left(-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}\right)$$

$$> \frac{1}{2}\left(-(\sigma+1) + (\sigma+1)\right) = 0 \qquad (\text{Positive})$$

By stable manifold theorem, $(0, 0, 0)$ is unstable.

Symmetry of the Lorenz System

- Claim: If $(x(t), y(t), z(t))$ solves the Lorenz system,

  then $(-x(t), -y(t), z(t))$ solves it as well.

- Proof: $(-x)' = -x' = -(\sigma(y-x)) = \sigma((-y)-(-x))$.

  $(-y)' = -y' = -(\rho x - y - xz) = \rho(-x) - (-y) - (-x)z$

- Consequence: When study the stability of $\vec{x}_{+,-}$, it suffices to consider one of them

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- Consider the case: $\sigma = 10$, $\rho = 28$, $\beta = 8/3$.

$$\begin{cases} x' = 10(y-x) \\ y' = 28x - y - xz \\ z' = xy - \frac{8}{3}z \end{cases}$$

$\Rightarrow$ Equilibrium points: $\vec{x_e} = (0,0,0)$,

$\vec{x}_+ = (6\sqrt{2}, 6\sqrt{2}, 27)$

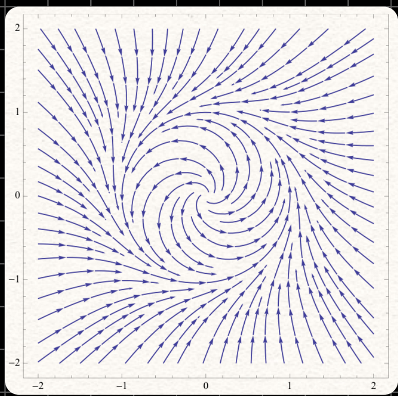
$\vec{x}_- = (-6\sqrt{2}, -6\sqrt{2}, 27)$

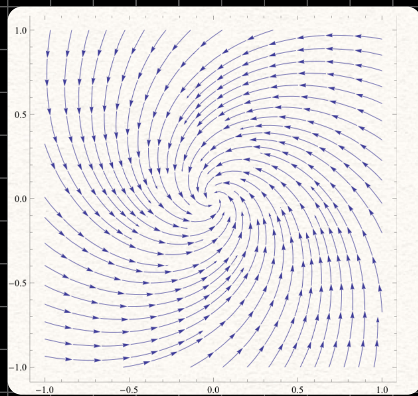
$$DL = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28-z & -1 & -x \\ y & x & -\frac{8}{3} \end{bmatrix}$$

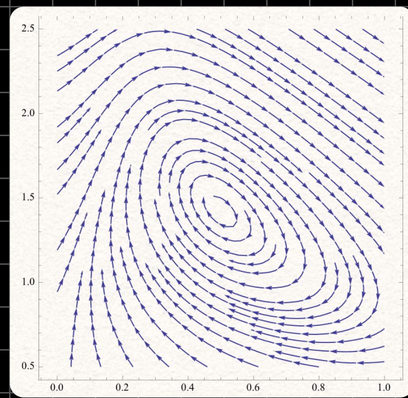
$$DL_{\vec{x}_+} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3} \end{bmatrix} \Rightarrow \begin{aligned} \lambda_1 &\approx -13.85 \\ \lambda_2 &\approx 0.09 + 10.19i \\ \lambda_3 &\approx 0.09 - 10.19i \end{aligned}$$

- ## Attractor

- ### "Motivation"







Many dynamical systems have dissipative behaviors.

(Consider dissipation from friction, heat lost, resistence, etc.)

$\Rightarrow$ Moving to places with minimum "energy".

Got attracted by "attractor".

- **Def.** (Attractor) [Lecture version]

Let φ_t be the flow map to the system in $\mathbb{R}^d$.

A set $\Lambda \subset \mathbb{R}^d$ is an attractor if :

① Λ is compact and invariant.

$\forall \vec{x}_0 \in \Lambda,$
$\varphi_t(\vec{x}_0) \in \Lambda$

② Exist an open neighbourhood $\mathcal{U}$ of Λ, s.t

$\forall \vec{x} \in \mathcal{U}, \quad \varphi_t(\vec{x}) \in \mathcal{U}$ for all $t \geq 0$ and ($\mathcal{U}$: Basin of attraction)

$\bigcap_{t \geq 0} \varphi_t(\mathcal{U}) = \Lambda$.

③ $\forall \vec{y}_1, \vec{y}_2 \in \Lambda, \exists$ open neighbourhood $\mathcal{U}_i \subset \mathcal{U}$,

containing $\vec{y}_i$, s.t.

$\exists \vec{z}_1 \in \mathcal{U}_1$ and $\varphi_t(\vec{z}_1) \in \mathcal{U}_2$ for some $t > 0$.

- **Remark** : There's no common definition for attractors.

- Def. (Attractor) [Other version]

Let φ_t be the flow map to the system in $\mathbb{R}^d$.

A set $\Lambda \subset \mathbb{R}^d$ is an attractor if :

① Λ is compact and invariant.

② Exist an open neighbourhood $\mathcal{U}$ of Λ, satisfies :

for any open neighbourhood N of Λ, $\exists\ T > 0$, s.t

$\forall\ \vec{x} \in \mathcal{U}$, $\varphi_t(\vec{x}) \in N$ for $t > T$.

③ There's no proper subset of Λ satisfies ① and ②.

- <u>Introduction to Bifurcation theory</u>

Sometimes, the behavior of ODE systems may depend on some parameters.

e.g. ρ in the Lorenz system.

$\left.\begin{array}{l}\text{The number of equilibrium points} \\ \text{The stability of equilibrium points}\end{array}\right\}$ depends on the value of ρ.

Bifurcation theory: The study of structural change of Dynamical System due to parameters.

- Examples: (Focus on 1D systems, easy to analyze stability.)

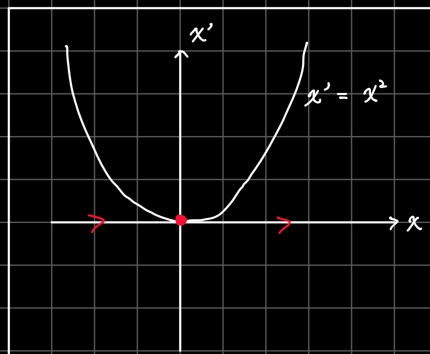
① $x' = x^2 - a = f_a(x)$

  - Equilibrium points: $x_1 = \sqrt{a}$ and $x_2 = -\sqrt{a}$

If $a > 0$, 2 equilibrium points.

If $a = 0$, 1 equilibrium points.

If $a < 0$, 0 equilibrium points.



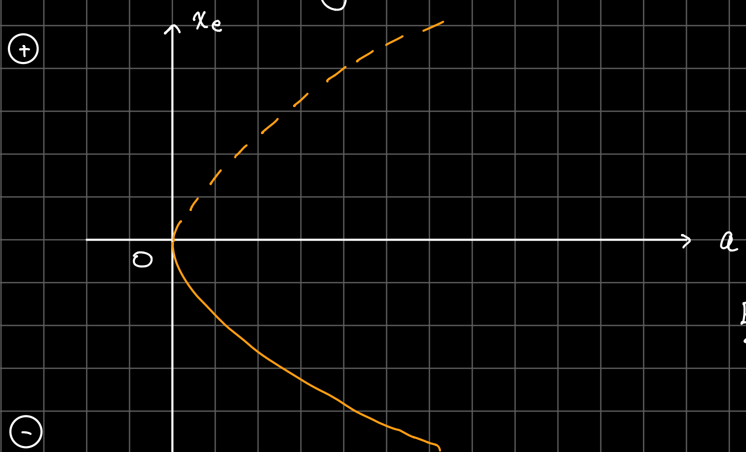
(Phase Portrait, $a = 0$)

Consider $f_a'(x) = 2x$.

When $a > 0$, $x_1 = \sqrt{a} > 0$ is unstable
$x_2 = -\sqrt{a} < 0$ is stable

When $a = 0$, $x_e = 0$ is a saddle point. (unstable)

- <u>Bifurcation Diagram</u>: (Saddle-node Bifurcation)



Solid line: Stable
Dash line: Unstable

Equilibrium points collide and annihilate.

② $x' = ax - x^2 = (a - x)x$

- Equilibrium points : $x_1 = a$ and $x_2 = 0$

If $a > 0$, 2 equilibrium points.

If $a = 0$, 1 equilibrium points.

If $a < 0$, 2 equilibrium points.

Consider $f_a'(x) = a - 2x$.

- $a > 0$:

(i) $x = a$: $f_a'(a) = -a < 0$ (stable)

(ii) $x = 0$: $f_a'(0) = a > 0$ (unstable)

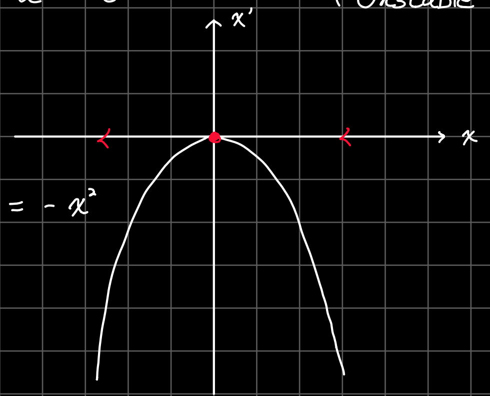
- $a < 0$: } stability changes

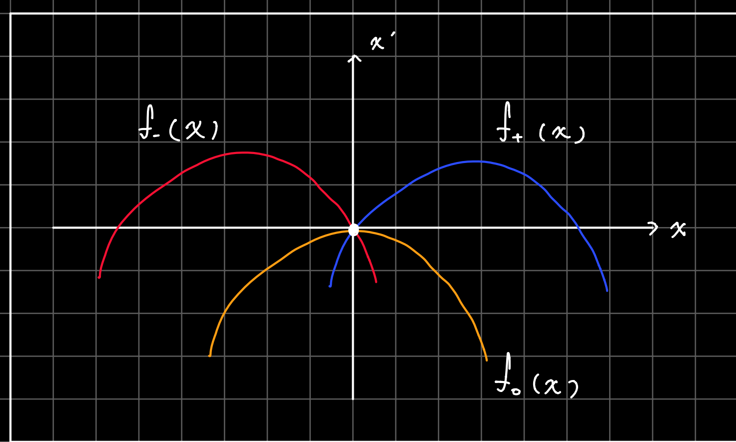
(i) $x = a$: $f_a'(a) = -a > 0$ (unstable)

(ii) $x = 0$: $f_a'(0) = a < 0$ (stable)

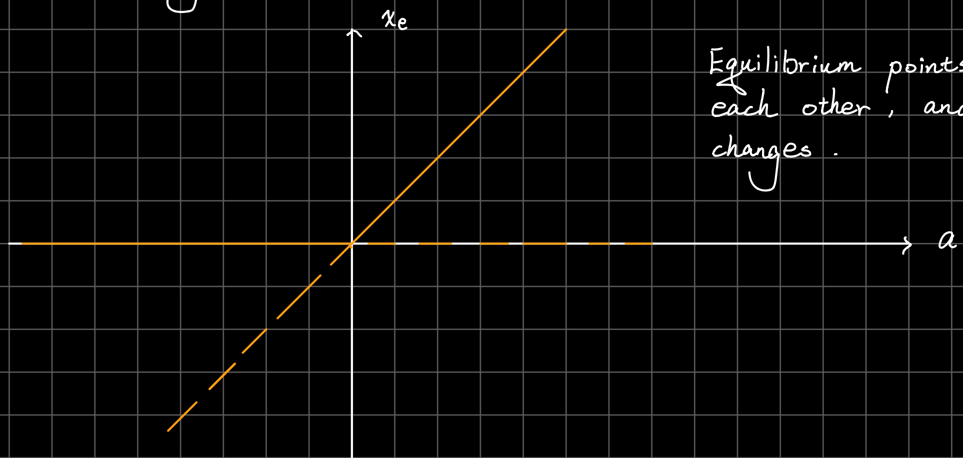
- $a = 0$: (Unstable)

$f_0(x) = -x^2$





- Bifurcation Diagram : (Transcritical Bifurcation)

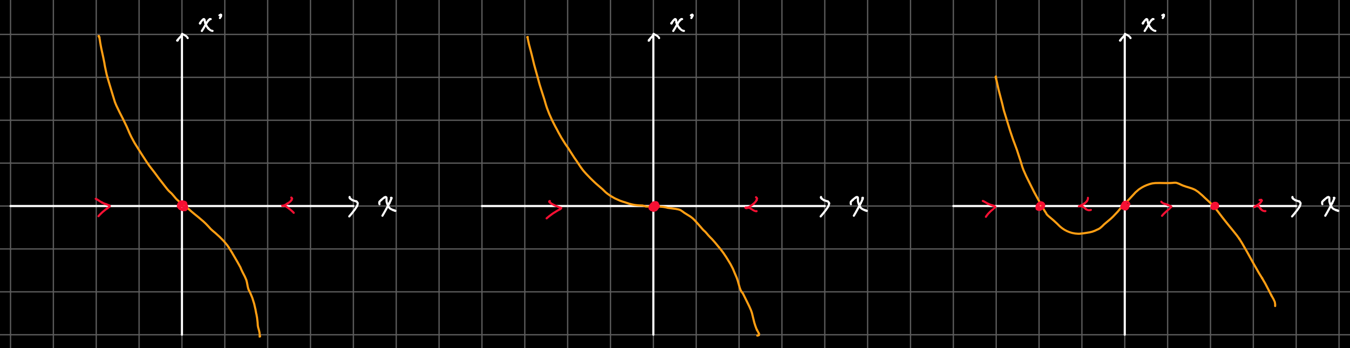


Equilibrium points pass through each other, and their stability changes.

(3) $x' = ax - x^3 = (a - x^2)x$

Equilibrium points : $x_1 = 0$, $x_\pm = \pm\sqrt{a}$

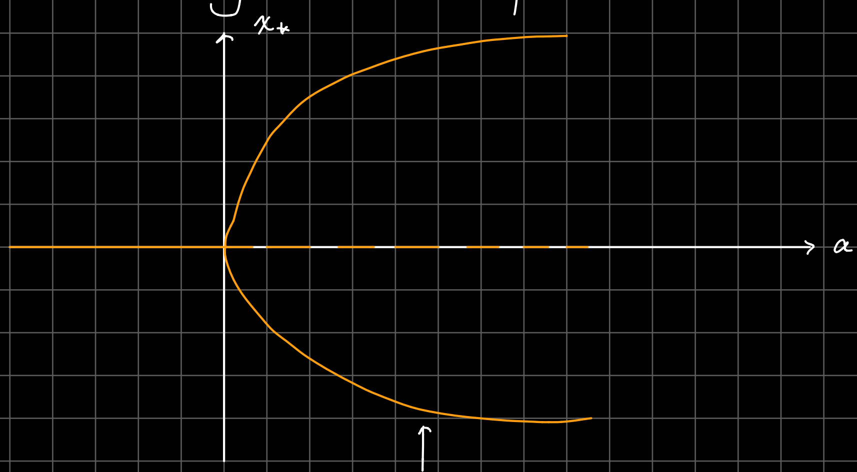
- Phase Portraits :

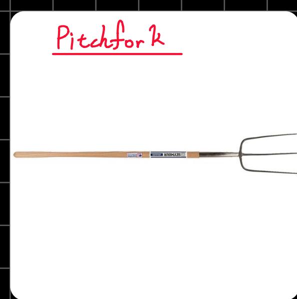


$(a < 0)$ $(a = 0)$ $(a > 0)$

- x_1 is stable for $a \leq 0$, unstable for $a > 0$
- $x_\pm$ are stable for $a > 0$.

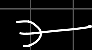
- Bifurcation Diagram : (Supercritical Pitchfork Bifurcation)

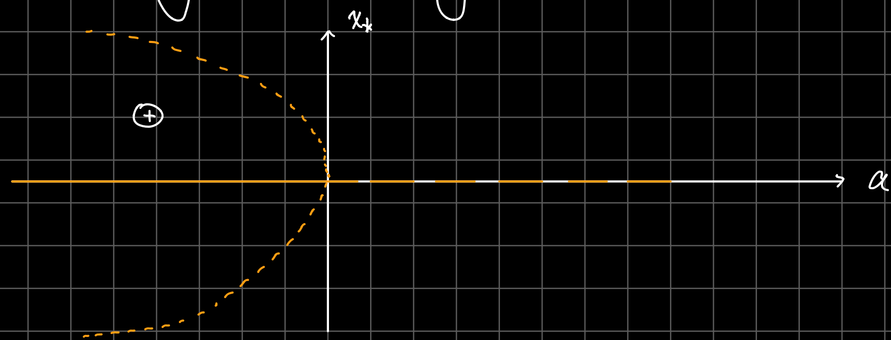




- One may consider : $x' = ax + x^3$.

which gives you the subcritical pitchfork bifurcation.

with the following bifurcation diagram : Pitchfork like this : 



- Other types of Bifurcations exist, e.g. Hopf Bifurcation.

The general theory of Bifurcation applies to $\vec{x}' = \vec{F}_a(\vec{x})$.

The examples above are some typical models only, for local bifurcation.

We also have global bifurcation, which is more difficult to study.

- Many texts and videos can be found on this topic.

- END -

THANK FOR ATTENDING THE TUTORIALS.

GOOD LUCK !

If you have any question about the course, feel free to contact me.

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