Lecture 23

12/05/2020

Definition :

$x' = F(x)$ in $\mathbb{R}^d$, φ_t = flow map.

Λ is said to be an <u>attractor</u> if

(1) Λ is compact and <u>invariant</u> $\leftarrow$ $x_0 \in A$, then $\varphi_t(x_0) \in \Lambda$ $\forall t \in (-\infty, \infty)$.

(2) $\exists\, \mathcal{U}$ open. $\mathcal{U} \supset \Lambda$

and $\varphi_t(\mathcal{U}) \subset \mathcal{U}$ $\forall t \geq 0$,

and $\Lambda = \bigcap_{t \geq 0} \varphi_t(\mathcal{U})$.

(3) $\forall y_1 \neq y_2$ in Λ, $\exists\, \overset{\ni y_1}{\mathcal{U}_1}, \overset{\ni y_2}{\mathcal{U}_2} \subset \mathcal{U}$

s.t. any solution from $\mathcal{U}_1$ $\quad \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

e.g. · {equilibrium point}.

· periodic solution.

· the torus in HW4.

· $\omega(y_0) \ni z_0$

$\Rightarrow z_0 = \lim_{n \to \infty} \varphi_{t_n}(y_0)$

Condition (2) will pass through U_2 at later time.

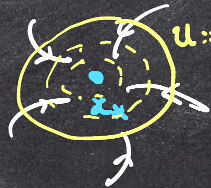
$\underbrace{\varphi_t(x_0) = \lim_{n\to\infty} \varphi_{t+t_n}(y_0)}_{\in\, \omega(y_0)}$

e.g. $\{\underbrace{\text{equilibrium point}}_{x_*}\} =: \Lambda$

If x_* is a saddle, $\longrightarrow$ $\varphi_t(U) \not\subset U.$

$\otimes$

If DF_* has negative eigenvalues, $\longrightarrow$ $U := \{Q < c\}.$



then $\varphi_t(U) \subset U.$

Claim:

$\bigcap_{t\geq 0} \varphi_t(U) = \{x_*\}.$

$\supset$ trivial.

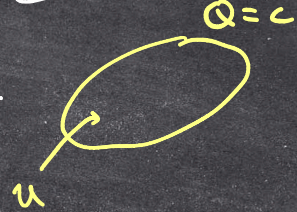
Proof: If $y \in \bigcap_{t\geq 0} \varphi_t(U)$,

Our proof of Poincaré-Lyap.

$\frac{d}{dt} \underbrace{Q(x,y)}_{\text{positive quadratic form.}} \leq -\mu Q(x,y)$ near x_*

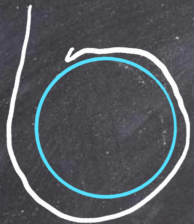
then $\forall t \geq 0$, $\exists\, u(t) \in \mathcal{U}$ s.t. $y = \underline{\varphi_t(u(t))}$.

$$|y| = |\varphi_t(u(t))| \leq Ce^{-\mu t} \quad \forall t \geq 0.$$



$\Rightarrow y = x_*$.

$Q(\varphi_t(u)) \leq Ce^{-\mu t}$.

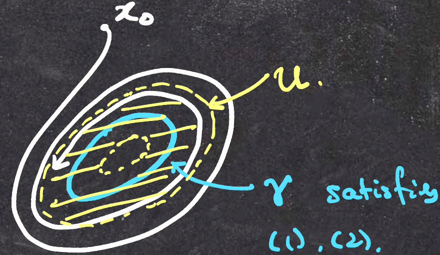


__e.g.__ Poincaré-Bendixson:

__Claim__: $\bigcap_{t \geq 0} \varphi_t(\mathcal{U}) = \gamma$ $\supset$ trivial.

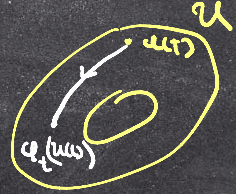
__Proof__: $\lim_{t \to +\infty} d(\varphi_t(u), \gamma) = 0$.

$\forall u \in \mathcal{U}$ (from the proof of $\omega(x_0) = \gamma$)



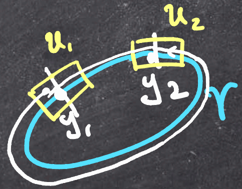
If $y \in \bigcap_{t \geq 0} \varphi_t(\mathcal{U})$, then $\forall t \geq 0$, $\exists\, u(t) \in \mathcal{U}$ s.t. $y \in \varphi_t(u(t))$.

Exercise : $d(y, \gamma) = 0$.



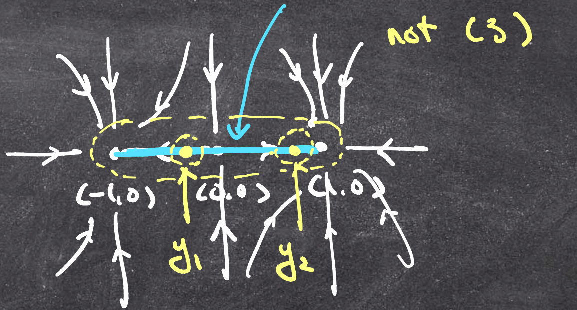
Condition (3)

e.g.



Counter-example of (3) :

$$\begin{cases} x' = x - x^3 \\ y' = -y. \end{cases}$$

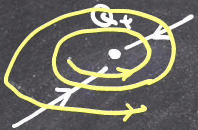


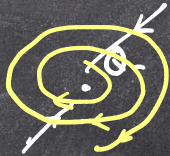
Guckenheimer, Williams → Model Lorenz system.

Tucker proved that the model corresponds qualitatively to the Lorenz system

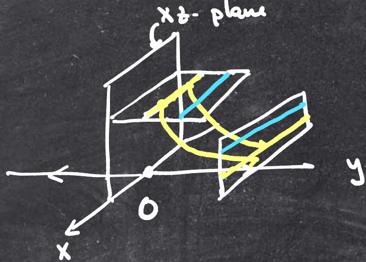
Model Lorenz system.

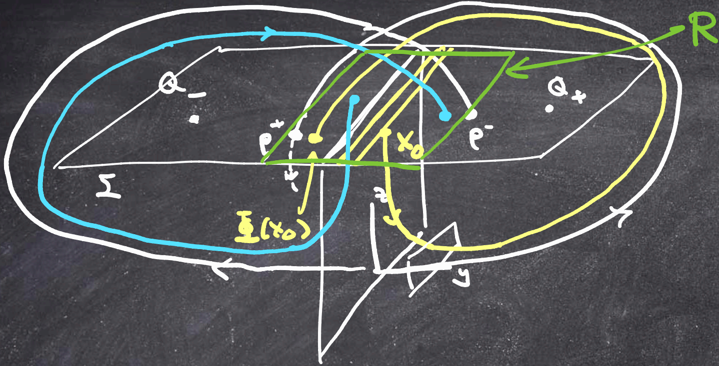
eigenvalues: +, +, −





near (0.0.0)





Poincaré map: $\Phi : \Sigma \to \Sigma$.

$x_0 \longmapsto$ the first point $\varphi_t(x_0)$ intersects Σ. $t>0$

$$A := \bigcap_{n \geq 0} \overline{\Phi^n(R)}$$

$$\Lambda := \bigcup_{t \geq 0} \varphi_t(A) \cup \{(0,0,0)\}$$

Claim: Λ is an attractor.