Lecture 22

07/05/2020

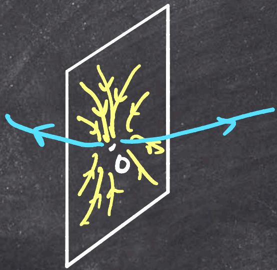
$\boxed{r > 1}$ $\boxed{\sigma > b+1}$

$$DL(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

eigenvalues $\lambda = \underbrace{-b}_{-}, \ \frac{1}{2}\left(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 \underbrace{-4\sigma(1-r)}_{+}}\right)$

$+$

$-$



$$Q_\pm = \left(\pm\sqrt{b(r-1)}\,,\ \pm\sqrt{b(r-1)}\,,\ r-1\right) \qquad \boxed{r>1}$$

Direct computation $\Rightarrow$ $\det\left(D\mathcal{L}(Q_\pm) - \lambda I\right) = 0$

$$\Leftrightarrow \underbrace{\lambda^3 + (1+b+\sigma)\lambda^2 + b(\sigma+r)\lambda + 2b\sigma(r-1)}_{p(\lambda)} = 0.$$

$\lambda \geq 0$ $\Rightarrow$ $p(\lambda) \geq 2b\sigma(r-1) > 0$.

$\Rightarrow$ all real roots of $p(\lambda)$ must be negative.

$r=1$: eigenvalues

$$\lambda^3 + (1+b+\sigma)\lambda^2 + b(\sigma+1)\lambda = 0.$$

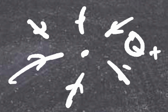
$$\Rightarrow \lambda = 0,\ -b,\ -\sigma-1. \qquad \sigma > b+1$$

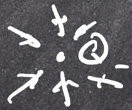
distinct.

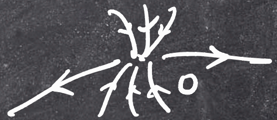
near $r \approx 1$: eigenvalues are all real.

Conclusion:

$\exists \delta > 0$ s.t. $\underline{1 < r < \overbrace{\delta + 1}^{=\sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)}} \Longrightarrow Q_\pm$ is asymptotically stable.







From now. take $r \gg 1$.

A is invariant $\overset{\text{def}}{\Leftrightarrow} \left(y \in A \implies \varphi_t(y) \in A \quad \forall t \in (-\infty, \infty) \right)$.

$\omega(x_0)$ is an invariant set

<u>Proof</u>: $y \in \omega(x_0) \implies y = \lim_{n\to\infty} \varphi_{t_n}(x_0)$

$$\implies \varphi_s(y) = \lim_{n\to\infty} \varphi_{s+t_n}(x_0) \in \omega(x_0).$$

$$\forall s \in \mathbb{R}.$$

closed

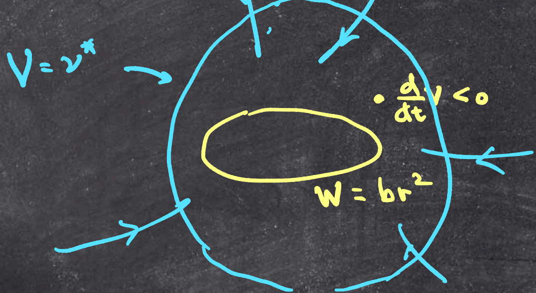
<u>Goal</u>: Search for invariant set A for the Lorenz system.

$x_0 \in A \implies \omega(x_0) \subset A$.

Claim: $\exists$ ellipsoid $\underbrace{rx^2 + \sigma y^2 + \sigma(z-2r)^2}_{V(x,y,z)} = \underset{\text{const.}}{\nu^*}$

s.t. any solution to the Lorenz system must eventually enter the $\{V = \nu^*\}$ and stay there for all future time.

Proof: $$\frac{d}{dt}V(\vec{x}(t)) = -2\sigma\Big(\underbrace{rx^2 + y^2 + b(z-r)^2}_{W(x,y,z)} - br^2\Big)$$



Cor: $\forall x_0 \in \mathbb{R}^3$
$\omega(x_0) \subset \{V \leq \nu^*\}$.

Claim:

$D_t := \varphi_t(D_0)$. Then $\mathrm{vol}(D_t) \to 0$ as $t \to +\infty$.

Cor:

A $\to$ 

invariant.

$\Downarrow$

$\mathrm{vol}(A) = 0$.

$\varphi_t(D_0) \supset A$

$\mathrm{vol}(\varphi_t(D_0)) \geq \mathrm{vol}(A)$

$\downarrow$

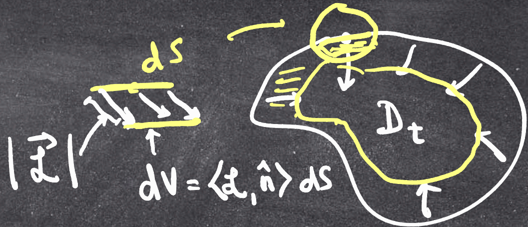
0 as $t \to \infty$.

In contrast: in $\mathbb{R}^2$.

$\gamma \cup D$ invariant.

Proof of claim

$$\frac{d}{dt}\,\mathrm{vol}(D_t) = \int_{\partial D_t} \langle \mathcal{L}, \hat{n} \rangle \, dS$$



$$= \int_{D_t} \mathrm{div}\,\mathcal{L}\; dV$$

$$\Downarrow$$

$$= \int_{D_t} -\underbrace{(\sigma + 1 + b)}_{\oplus}\, dV = -(\sigma+1+b)\,\mathrm{vol}(D_t).$$

$$\Rightarrow \quad \mathrm{vol}(D_t) = \mathrm{vol}(D_0)\cdot e^{-(\sigma+1+b)t} \longrightarrow 0.$$

as $t \to \infty$.