Lecture 21

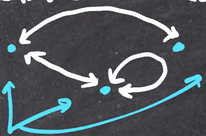
05/05/2020

- 

$\Rightarrow \exists$ periodic solution γ and $\gamma = \omega(x_0)$.

In $\mathbb{R}^2$, possible $\omega(x_0)$ for a C^1-system $\vec{x}' = F(\vec{x})$:

- $\omega(x_0) = \{\text{point}\}$, or
- $\omega(x_0) =$ periodic solutions, or
- trajectories "connecting" equilibrium points, or



equilibrium point

- union of all these.

i.e. no "chaotic behavior" in $\mathbb{R}^2$.

- sensitive dependence on initial data.
- complicated limit / invariant set.

Consider $\mathbb{R}^4$:

$$\begin{cases} x'' = -\omega_1^2 x \\ y'' = -\omega_2^2 y \end{cases} \Rightarrow (*) \begin{cases} x_1' = x_2 \\ x_2' = -\omega_1^2 x_1 \\ y_1' = y_2 \\ y_2' = -\omega_2^2 y_1 \end{cases}$$

$$\vec{x}(t) = \underbrace{P[\quad]P^{-1}}_{e^{tA}}\vec{x}_0$$

$$P^{-1}\vec{x}(t) = [\quad]P^{-1}\vec{x}_0$$

$\exists$ linear $T: \mathbb{R}^4 \to \mathbb{R}^4$ s.t.
$\qquad\qquad \shortparallel \quad\;\; \shortparallel$
$\qquad\qquad \mathbb{C}^2 \quad \mathbb{C}^2$

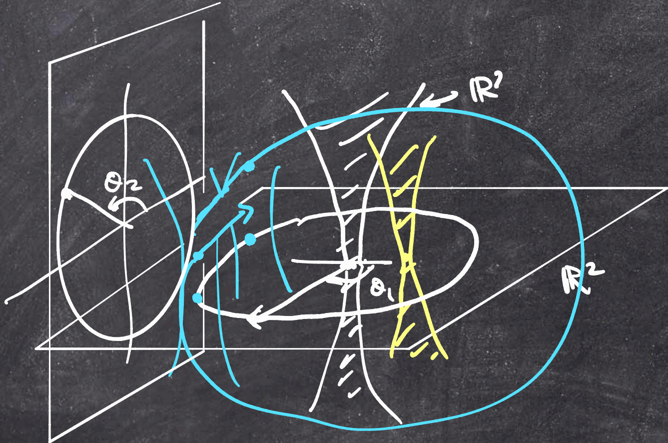
$$T(\underbrace{\vec{x}(t)}_{\text{solution to } (*)}) = (\underbrace{r_1 e^{i(\tilde{\theta}_1 - \omega_1 t)}}_{\text{in } \mathbb{C} = \mathbb{R}^2}, r_2 e^{i(\tilde{\theta}_2 - \omega_2 t)})$$

(x_1, x_2, y_1, y_2)
$= (x_1 + i x_2, y_1 + i y_2)$

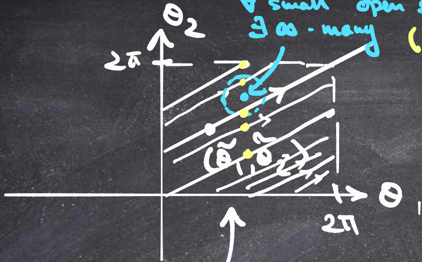
HW4.

HW1: If $\boxed{\frac{\omega_1}{\omega_2} \notin \mathbb{Q}}$, then (*) has no non-trivial periodic solution.

Ask: $\omega(\vec{x}_0) = ?$ $\quad T(\omega(\vec{x}_0)) = ?$



$\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2.$



$\forall$ small open set around any point on

$\exists\ \infty$-many $(\cdot, \theta_1)$ in

fixed.

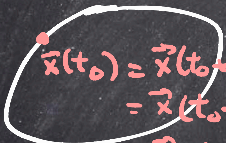
$$\frac{\omega_1}{\omega_2} \notin \mathbb{Q}.$$

$$T(\omega(\vec{x}_0)) = S^1 \times S^1. \quad \leftarrow \text{HW 4.}$$

(compared with 2043:

$\Rightarrow \operatorname{LIM}\{\sin(n)\} = [0, 1].$)

$$\vec{x}(T+t) = \vec{x}(t)$$



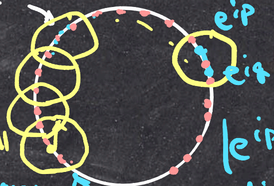
$$\vec{x}(t_0) = \vec{x}(t_0+T)$$
$$= \vec{x}(t_0+2T)$$
$$= \vec{x}(t_0+3T)$$

$$\vec{x}(t_0+nT) = \vec{x}(t_0) \quad \forall n$$

$$\downarrow$$

$$\vec{x}(t_0) \in \omega(\vec{x}_0).$$

$$\operatorname{Lim}\{e^{in}\} =$$



$$\left(e^{i(p-q)}\right)^k$$

$$|e^{ip} - e^{iq}| < \text{diam}\ \bigcirc$$

Lorenz system

E. Lorenz. $\neq$ L. Lorenz.

L. Lorenz

$\neq$ H. Lorentz.

PHYS

$$\left\{\begin{aligned} x' &= \sigma(y-x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}\right\} \mathcal{L}(\vec{x})$$

$\sigma, r, b > 0$ constants

$$\boxed{\sigma > b+1}$$

equilibrium points:

$$(0,0,0), \quad Q_{\pm} = \Big(\underbrace{\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1}_{r \geq 1}\Big)$$

$$D\mathcal{L} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

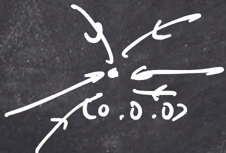
$$D\mathcal{L}(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & \boxed{-b} \end{bmatrix}$$

eigenvalues? $\quad (-\sigma-\lambda)(-1-\lambda)(-b-\lambda) + r(-b-\lambda) = 0$

$$\Rightarrow \lambda = -b,\ \frac{1}{2}\left(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}\right)$$

When $\boxed{r<1}$: all eigenvalues are <u>negative</u> real.

or have <u>negative</u> real part.



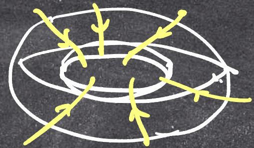
$$\left(V(x,y,z) = x^2 + \sigma y^2 + \sigma z^2 \right).$$

$\uparrow$

strict Lyapunov function for $\vec{0}$.

$\boxed{r<1}$ $\frac{d}{dt}V(\vec{x}(t)) \underset{\sim}{<} 0 \quad \forall \vec{x}(t) \neq \vec{0}$.

check.



Hirsch-Smale-Devaney. Ch. 14.