- Today's tutorial :
  1. Recall the Gradient (and Hamiltonian) Systems.
  2. Introduction to Periodicity of solutions to $\vec{x}' = \vec{F}(\vec{x})$.
  3. Example : Gradient System
  4. Bendixson - Dulac's Theorem
  5. Introduction to Poincare - Bendixson's Theorem.

- Last time, we studied the following systems :
  - Gradient System : $\vec{x}' = -\nabla f(\vec{x})$
    - $\frac{d}{dt} f(\vec{x}(t)) = -|\nabla f|^2 \leq 0$

  $\Rightarrow$ Possible candidate of Lyapunov function : $f(\vec{x}) - f(\vec{x}_*)$.

  Need $\vec{x}_*$ to be an isolated, local minimum of f !

  - Hamiltonian System : $\begin{cases} p' = -\frac{\partial H}{\partial q} \\ q' = \frac{\partial H}{\partial p} \end{cases}$

    - $\frac{d}{dt} H(p, q) = H_p \cdot p' + H_q \cdot q' = 0$

  $\vec{x} = \langle p, q \rangle$

  $\Rightarrow$ Possible candidate of Lyapunov function : $H(\vec{x}) - H(\vec{x}_*)$

  Need $\vec{x}_*$ to be an isolated, local minimum of H !

• Some motivations to Chapter 04

Recall that :

if $\vec{x}_*$ is an equilibrium point of $\vec{x}' = \vec{F}(\vec{x})$.

$\Rightarrow$ $\vec{x}_*$ is a stationary solution to the system.

i.e. $\varphi_t(\vec{x}_*) = \vec{x}_*$ for any $t \in \mathbb{R}$

Or we may write it as

$$\varphi_t(\vec{x}_*) = \varphi_{t+T}(\vec{x}_*) \quad \text{for any } t \in \mathbb{R},\ T \in \mathbb{R}.$$

A special case of periodic solution!

Consider the gradient system : $\vec{x}' = -\nabla f(\vec{x})$

If it has a periodic solution $\vec{x}(t)$, with period $T > 0$.
($\vec{x}(t) = \vec{x}(t+T)$ for any $t \in \mathbb{R}$)

then we have

$$
\begin{aligned}
0 &= f(\vec{x}(t+T)) - f(\vec{x}(t)) \\
&= \int_t^{t+T} \frac{d}{ds} f(\vec{x}(s))\, ds \\
&= -\int_t^{t+T} \underbrace{|\nabla f(\vec{x}(s))|^2}_{\geq 0}\, ds
\end{aligned}
$$

$\Rightarrow$ $\nabla f(\vec{x}) \equiv \vec{0}$ on $[t, t+T]$, / for all $t \in \mathbb{R}$.

$\Rightarrow$ $\vec{x}(t) = \vec{x}_*$

• Conclusion : The only periodic Solution is the equilibrium point.

- A More General Result.

Exercise 4.2. Let $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 vector field. Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a C^1 scalar function such that for any solution curve $\mathbf{x}(t)$ of the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, we have

$$\frac{d}{dt} f(\mathbf{x}(t)) \le 0 \quad \text{for any } t \in \mathbb{R}.$$

Prove that any periodic solution (if there is any) must lie on a level set of f.

Ans : Note that : $\frac{d}{dt} f(\vec{x}(t)) = \nabla f \cdot \vec{x}' = \nabla f \cdot \vec{F} \le 0 \quad \forall t \in \mathbb{R}$

If there exists a periodic Solution $\vec{x}(t)$, with periodic solution.

Then $f(\vec{x}(t)) = f(\vec{x}(t+T)) \quad \forall t \in \mathbb{R}$

$\Rightarrow \frac{d}{dt} f(\vec{x}(t)) \equiv 0$ along this solution curve $\vec{x}(t)$.

$\Rightarrow \nabla f \cdot \vec{F} \equiv 0$ along this solution curve $\vec{x}(t)$.

Since ∇f is the normal vector to the level set of f.

$\vec{F}$ always tangent to the level set of f.

$\Rightarrow \vec{x}(t)$ must lie on the level set of f. #

- Bendixson - Dulac Theorem

Theorem 4.6 (Bendixson-Dulac's Theorem). *Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field on $\mathbb{R}^2$. Denote the components of $\mathbf{F}$ by:*

$$\mathbf{F}(x, y) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix}.$$

Dulac Function

If there exists a C^1 scalar function $h(x, y) : \Omega \to \mathbb{R}$ defined on a simply-connected region $\Omega \subset \mathbb{R}^2$, such that

$$\frac{\partial(hF_1)}{\partial x} + \frac{\partial(hF_2)}{\partial y} = \nabla \cdot (h\vec{F}) > 0$$

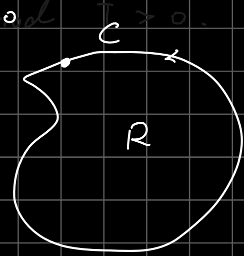
is positive in Ω, then the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ does not have any non-trivial periodic solution inside Ω.

- Proof : (By contradiction)

Suppose $\vec{x}(t)$ is a nontrivial periodic Sol. with period $T > 0$

then it forms a closed orbit, denoted as C

Also, denote the region enclosed by C to be R



By Green's Theorem,

$$\iint_R \underbrace{\frac{\partial(hF_1)}{\partial x} + \frac{\partial(hF_2)}{\partial y}}_{>0} \, dA = \iint_R \frac{\partial(hF_1)}{\partial x} - \frac{\partial(-hF_2)}{\partial y} \, dA$$

$$= \oint_C (-hF_2)\, dx + (hF_1)\, dy > 0$$

But note that $dx = x' \, dt = F_1 \, dt$

$dy = y' \, dt = F_2 \, dt$

$$\Rightarrow \oint_C \underbrace{(-hF_2 \cdot F_1 + hF_1 \cdot F_2)}_{=0} \, dt = 0$$

Contradiction #

- Poincaré - Bendixson's Theorem

Theorem 4.8 (Poincaré-Bendixson's Theorem)**.** *Let "$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$" be a C^1 vector field in $\mathbb{R}^2$ and consider the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Suppose K is a set in $\mathbb{R}^2$ such that:*

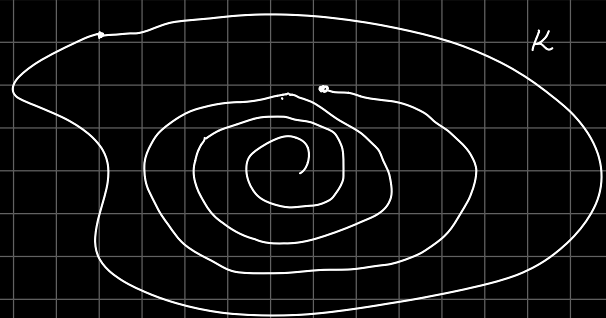
(1) K is closed and bounded;

(2) the system has no equilibrium point in K; and

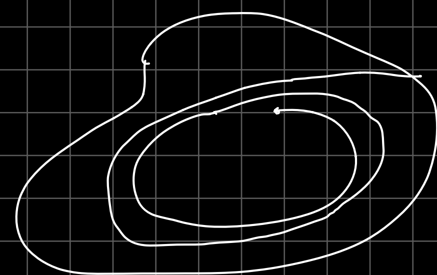
(3) K contains a forward trajectory of the system, i.e. there exists $\mathbf{x}_0 \in K$ such that $\varphi_t(\mathbf{x}_0) \in K$ for any $t \geq 0$. Here φ_t denotes the flow of the system.

Then, the system has a non-trivial closed orbit in K.

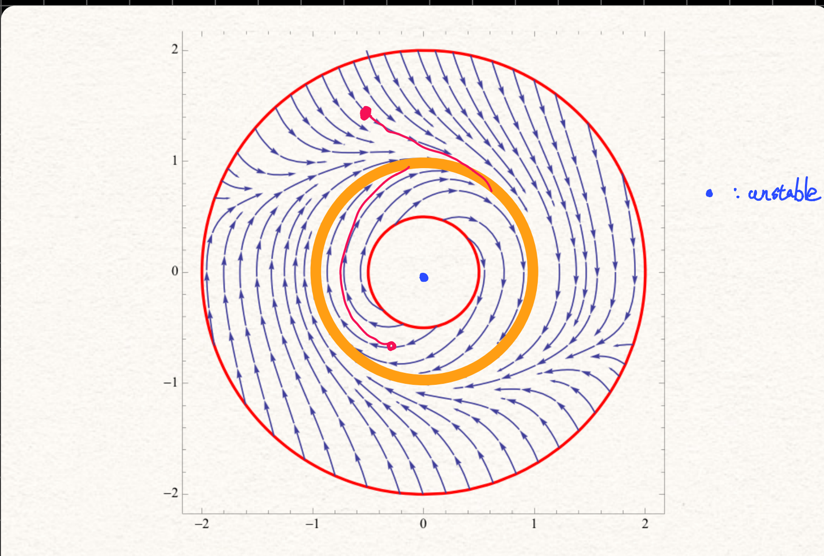
If $\vec{y}(t)$ is a bounded solution to the system, and it does not appraching any equilibrium point.



K, not $\vec{x}_*$ inside K.

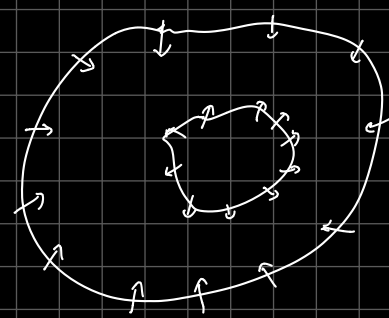


not possible by the given condition



• : unstable

Figure 4.4. The solution curves inside the trapping region K in Example 4.4.



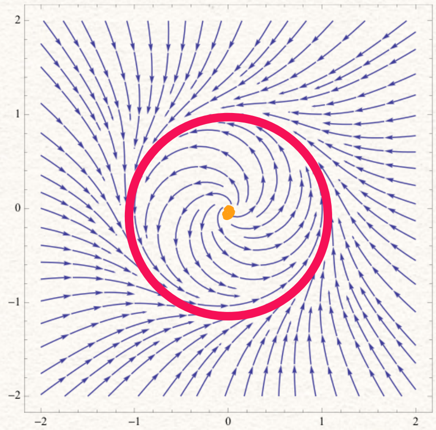


Figure 4.1. The phase portrait of the system in Example 4.1: $r' = r(1 - r^2), \quad \theta' = 1$