Lecture 16

14/04/2020

---

Gradient and Hamiltonian systems.

$$\vec{x}' = -\nabla f(\vec{x}) \quad \text{gradient system.}$$

$f: \mathbb{R}^d \to \mathbb{R}$, $C^2$

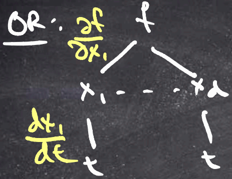
<u>Claim</u>: On a gradient system $\vec{x}' = -\nabla f(\vec{x})$, then

$$\frac{d}{dt} f(\underbrace{\vec{x}(t)}_{\text{solution}}) = -|\nabla f(\vec{x}(t))|^2 \leq 0$$

Condition (3) of Lyapunov function.

<u>Proof</u>: $\frac{d}{dt} f(\vec{x}(t)) = \langle \nabla f(\vec{x}(t)), \underbrace{\frac{d}{dt}\vec{x}(t)}_{-\nabla f} \rangle$ $\quad f \to x \to t$

$$= -|\nabla f(\vec{x}(t))|^2 \quad \square$$




$$\frac{d}{dt} f(\vec{x}(t)) = \sum_{j=1}^{d} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = -\sum_{j=1}^{d} \frac{\partial f}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = -|\nabla f|^2$$

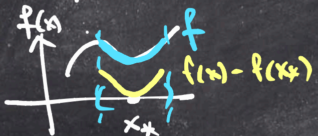
$$\vec{x}\,' = -\nabla f \iff \frac{dx_i(t)}{dt} = -\frac{\partial f}{\partial x_i}$$

On a gradient system $\vec{x}\,' = -\nabla f(\vec{x})$

$\vec{x}_*$ equilibrium point $\iff \nabla f(\vec{x}_*) = \vec{0} \iff \vec{x}_*$ is a critical point of $f$.

If $\vec{x}_*$ is an isolated local minimum of $f$, $\Rightarrow f(x) - f(x_*)$ satisfies (1) and (2)

then $f(\vec{x}) - f(\vec{x}_*)$ is a strict Lyapunov function for $\vec{x}_*$.




($\Rightarrow x_*$ is asymptotically stable)

$$\frac{d}{dt} f(\vec{x}(t)) = -|\nabla f(\vec{x}(t))|^2 < 0$$

unless $\vec{x} = \vec{x}_*$.


no other critical point.

$\nabla f(x) \neq \vec{0} \quad \forall x \neq x^*$ (near $x^*$)

$\therefore f(x) - f(x_*)$ is a strictly Lyapunov function for $\vec{x}_*$.

e.g.

$$\begin{cases} x' = -2x(x-1)(2x-1) \\ y' = -2y \end{cases}$$

equilibrium points $(0,0)$, $(1,0)$, $(\frac{1}{2}, 0)$

Find $f(x,y)$ st:

$$\begin{cases} 2x(x-1)(2x-1) = \frac{\partial f}{\partial x} \longrightarrow g(x) = \int 2x(x-1)(2x-1)\,dx = x^2(x-1)^2 + C. \\ 2y = \frac{\partial f}{\partial y} \Rightarrow f(x,y) = y^2 + \underbrace{g(x)}_{\text{for some function } g(x).} \Rightarrow \frac{\partial f}{\partial x} = g'(x) \end{cases}$$

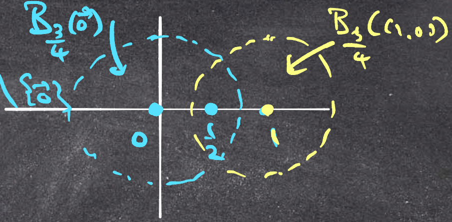
Take: $f(x,y) = x^2(x-1)^2 + y^2$

**Near** $(0,0)$:

$f(0,0) = 0$.

$f(x,y) > 0$ on $B_{\frac{3}{4}}(\vec{0}) \setminus \{\vec{0}\}$

$\therefore$ strict L.f. for $\vec{0}$




**Near** $(1,0)$:

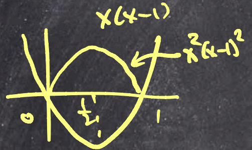
$f(1,0) = 0$

$f(x,y) > 0$ on $B_{\frac{3}{4}}((1,0)) \setminus \{(1,0)\}$

$\therefore$ strict L.f. for $(1,0)$.

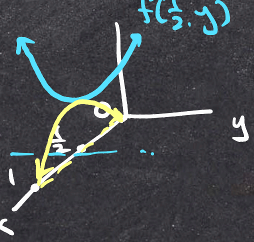
**Near** $(\frac{1}{2},0)$ ?

$$f(x,y) = x^2(x-1)^2 + \underbrace{y^2}.$$




$$f(\tfrac{1}{2}, y) = \frac{1}{16} + y^2.$$

$$f(x,0) = x^2(x-1)^2.$$




f is not a Lyapunov function for $(\frac{1}{2}, 0)$.

Given: $\vec{x}' = \vec{F}(\vec{x})$, How to tell whether $\exists f : \mathbb{R}^d \to \mathbb{R}$

s.t. $\vec{F} = -\nabla f$ ?

$\mathbb{R}^d$.

- $\vec{F} = (F_1, \dots, F_d) \iff \alpha = \sum_i F_i \, dx^i$

$$\vec{F} = -\nabla f \iff \alpha = -df$$

$$\exists f : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } \alpha = -df \iff d\alpha = 0.$$

$$\iff D\vec{F} \text{ is symmetric.}$$

- $\underset{C^2}{\exists f} : \mathbb{R}^d \to \mathbb{R}$ s.t. $\vec{F} = -\nabla f \iff D\vec{F}$ is symmetric.

$\left( \overset{d=3}{\iff} \nabla \times \vec{F} = \vec{0} \right).$

<u>Proof:</u> $(\Longrightarrow)$ $\quad F_i = -\dfrac{\partial f}{\partial x_i}$

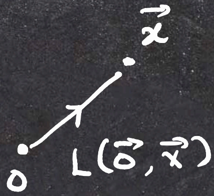
$$[DF]_{ij} = \frac{\partial F_i}{\partial x_j} = -\frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$[DF]_{ji} = -\frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{if } f \text{ is } C^2.$$

$(\Longleftarrow)$ Given $DF$ is symmetric.

$$\therefore \quad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall i, j.$$

Guess $\quad f(\vec{x}) := -\displaystyle\int_{L(\vec{0},\vec{x})} \vec{F} \cdot d\vec{r}.$




$$f(\vec{x}) = -\int_0^1 \vec{F}(t\vec{x}) \cdot \underbrace{\vec{x}\,dt}_{d\vec{r}}.$$

$\vec{r}(t) = t\vec{x}$

$0 \leq t \leq 1$

$$f(x_1,\dots,x_d) = -\int_0^1 \sum_{j=1}^{d} x_j F_j(tx_1,\dots,tx_d)\,dt.$$

$F_j$

$x_i$

$t$

**Claim:** $\dfrac{\partial f}{\partial x_i} = -F_j$

**Proof:**

$$\frac{\partial f}{\partial x_i} = -\int_0^1 \frac{\partial}{\partial x_i} \sum_{j=1}^{d} x_j F_j(tx_1,\dots,tx_d)\,dt$$

$$= -\int_0^1 \sum_{j=1}^{d} \Big(\delta_{ij} F_j(tx_1,\dots,tx_d) + t\,x_j \frac{\partial F_j}{\partial x_i}(tx_1,\dots,tx_d)\Big)\,dt$$

$$= -\int_0^1 F_i(tx_1,\dots,tx_d)\,dt - \sum_{j=1}^{d}\int_0^1 t\,x_j \underbrace{\frac{\partial F_j}{\partial x_i}(t\vec{x})}_{=\frac{\partial F_i}{\partial x_j}(t\vec{x})}\,dt$$

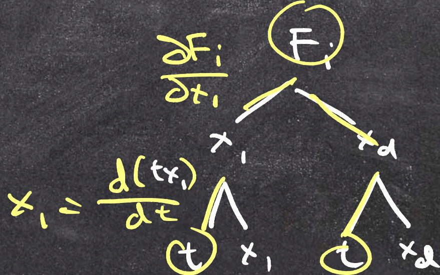
$$\sum_{j=1}^{d} \int_0^1 t x_j \frac{\partial F_j}{\partial x_i}(t\vec{x})\, dt = \int_0^1 \underbrace{\sum_{j=1}^{d} t x_j \frac{\partial F_i}{\partial x_j}(t\vec{x})}_{\substack{\shortparallel \\ t \frac{\partial}{\partial t} F_i(t x_1, \dots, t x_d)}} dt$$

$$= \int_0^1 \underbrace{t}_{u} \underbrace{\frac{\partial}{\partial t} F_i(t\vec{x})}_{v'} dt$$

$$= \Big[ t F_i(t\vec{x}) \Big]_{t=0}^{t=1} - \int_0^1 \underbrace{F_i(t\vec{x})}_{v} \underbrace{dt}_{du}$$

$$= F_i(\vec{x}) - \int_0^1 F_i(t\vec{x})\, dt$$

$$\therefore \frac{\partial f}{\partial x_i} = -F_i$$



$$\frac{dh}{dt} = \frac{d}{dt}\left(\frac{y^2}{2} + 1 - \cos x\right) = yy' + \sin x \cdot x'$$

$$= y\cdot(-\sin x) + \sin x \cdot y = 0.$$

---

Hamiltonian system: $\begin{cases} x' = -\dfrac{\partial H}{\partial y} \\ y' = \dfrac{\partial H}{\partial x} \end{cases}$ $\quad H: \mathbb{R}^2 \to \mathbb{R}.\quad C^2.$

let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Along $\vec{x}(t)$ any solution,

$$\frac{d}{dt} H(\vec{x}(t)) = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial H}{\partial x}\left(-\frac{\partial H}{\partial y}\right) + \frac{\partial H}{\partial y}\cdot\frac{\partial H}{\partial x}$$

$$= 0.$$

$\vec{x}' = \vec{F}(\vec{x})$ is Hamiltonian

$\Leftrightarrow$ $J\vec{F}$ is gradient.

$J\vec{F} = (-F_2, F_1)$

If $\nabla H(x_*) = 0$ and $x_*$ is an isolated local min. $\Rightarrow$ H is a Lyapunov function for $x_*$