Induction assumption:

$$\begin{cases} |x_j(t) - x_{j-1}(t)| \le \frac{1}{2^{j-1}} e^{-\mu t/2} |x(0)| \\ |y_j(t) - y_j(t)| \le \frac{1}{2^{j-1}} e^{-\mu t/2} |x(0)| \end{cases} \text{ for } j = 1, 2, \dots, n$$

- When $(\bar{x}, \bar{y}) \in B_\varepsilon(\bar{0})$, then $\|\nabla h(\bar{x}, \bar{y})\| < \delta := \frac{\mu}{4\sqrt{2}}$.

If

$$\underbrace{\left.\begin{cases} x_n(s), x_{n-1}(s) \\ y_n(s), y_{n-1}(s) \end{cases}\right\} \in \underline{\underline{B_\varepsilon(\bar{0})}}}_{s \in [0, \infty)} \Rightarrow \underbrace{\begin{cases} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2^n} e^{-\mu t/2} |x(0)| \\ |y_{n+1}(t) - y_n(t)| \le \frac{1}{2^n} e^{-\mu t/2} |x(0)| . \end{cases}}_{j = n+1 \text{ case.}} \quad \boxtimes$$

$$x_{n-1}(t) = \cancel{x_0(t)}^{\,0} + \sum_{j=1}^{n-1} \left(x_j(t) - x_{j-1}(t) \right)$$

$$|x_{n-1}(t)| \le \sum_{j=1}^{n-1} |x_j(t) - x_{j-1}(t)| \le \underbrace{\sum_{j=1}^{n-1} \frac{1}{2^{j-1}}}_{\le 2} \underbrace{e^{-\mu t/2}}_{\le 1} |x(0)|$$

$$\le 2|x(0)|.$$

Similarly: $|x_n(t)|, |y_{n-1}(t)|, |y_n(t)| \le 2|x(0)|$.

<u>Choose</u> $x(0)$ s.t. $2|x(0)| < \varepsilon$

ε s.t.

$(\bar{x}, \bar{y}) \in B_\varepsilon(0)$

$\Rightarrow |\nabla h_1|, |\nabla h_2| < \frac{\mu}{4\sqrt{2}}.$

So far:

$$\begin{cases} |x_n(t) - x_{n-1}(t)| \le \frac{1}{2^{n-1}} e^{-\mu t/2} |x(0)| & \forall n \in \mathbb{N} \\ |y_n(t) - y_{n-1}(t)| \le \frac{1}{2^{n-1}} \underbrace{e^{-\mu t/2} |x(0)|}_{\le 1.} & \forall t \in [0,\infty) \end{cases}$$

Weierstrass M-test

$$\Rightarrow \quad x_n(t) \rightrightarrows x_\infty(t) \quad \text{on } [0,\infty)$$
$$y_n(t) \rightrightarrows y_\infty(t)$$

$$F(x,y) = \begin{bmatrix} -\lambda x \\ \mu y \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$x_n(t) = e^{-\mu t}\left(x(0) + \int_0^t e^{\mu s} h_1(x_{n-1}(s), y_{n-1}(s))\, ds \right) \qquad n \ge 1.$$

$$y_n(t) = -e^{\lambda t} \int_t^\infty e^{-\lambda s} \underbrace{h_2(x_{n-1}(s), y_{n-1}(s))}_{\text{bounded} \le M.}\, ds.$$

$h_1 : \overline{B_\varepsilon(0)} \to \mathbb{R}$ continuous $\Rightarrow$ h_1 is uniformly continuous on $\overline{B_\varepsilon(0)}$.

$\downarrow$

$B_\varepsilon(0) \ni x_n \rightrightarrows x_\infty$

$B_\varepsilon(0) \ni y_n \rightrightarrows x_\infty$ $\quad\longrightarrow\quad$ $h_1(x_n(s), y_n(s)) \rightrightarrows h_1(x_\infty(s), y_\infty(s))$ on $s \in [0,\infty) \supset [0,t]$

$\downarrow$

$e^{\mu s} \le e^{\mu t}$ on $s \in [0,t]$.

$\hookrightarrow$ $e^{\mu s} h_1(x_n(s), y_n(s))$

$\rightrightarrows e^{\mu s} h_1(x_\infty(s), y_\infty(s))$

on $s \in [0,t]$.

$$\lim_{n\to\infty} x_n(t) = e^{-\mu t}\left(x(0) + \int_0^t \lim_{n\to\infty} e^{\mu s} h_1(x_n(s), y_n(s))\, ds \right)$$

$$x_\infty(t) = e^{-\mu t}\left(x(0) + \int_0^t e^{\mu s} h_1(x_\infty(s), y_\infty(s))\, ds \right)$$

$y_\infty(s)$

$$= -e^{\lambda t}\int_t^\infty e^{-\lambda s}\, h_2(x_\infty(s), y_\infty(s))\,ds$$

LDCT: $\begin{cases} |f_n(x)| \le g(x) \ \text{on } x \in E,\ \forall n \in \mathbb{N} & \leftarrow \text{indep. of } n \\ \int_E |g(x)| < \infty. \end{cases}$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n \overset{?}{=} \int_E \lim_{n\to\infty} f_n$$

Claim: $(x_\infty(t), y_\infty(t)) \to (0,0)$ as $t \to +\infty$.

Proof: $|x_n(t) - x_{n-1}(t)| \le \frac{1}{2^{n-1}} e^{-\mu t/2} \underbrace{|x(0)|}$ $\forall n \in \mathbb{N}$, $\forall t \in [0,\infty)$.

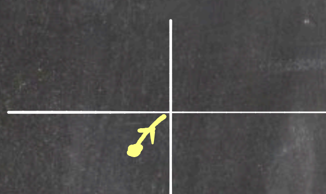
$$x_n(t) = \cancel{x_0(t)}^{\,0} + \sum_{j=1}^{n} (x_j(t) - x_{j-1}(t))$$

$$\underbrace{|x_n(t)|} \le \underbrace{\sum_{j=1}^{n} \frac{1}{2^{j-1}}}_{\le 2} e^{-\mu t/2} \underbrace{|x(0)|}_{<\varepsilon} < 2\varepsilon e^{-\mu t/2} \quad \forall n,\ \forall t \ge 0.$$

$$\Big\downarrow {\scriptstyle n\to\infty}$$

$$|x_\infty(t)| \le 2\varepsilon e^{-\mu t/2}$$

$\lim_{t\to+\infty} |x_\infty(t)| = 0.$ Similarly, $\lim_{t\to+\infty} |y_\infty(t)| = 0.$



(Stable part).

$F: \mathbb{R}^d \to \mathbb{R}^d$ is C^1.

$\vec{x}' = F(\vec{x})$, $\vec{F}(\vec{x}_*) = \vec{0}$.

If $\exists\, L: \underset{\ni x_*}{U} \xrightarrow{} \mathbb{R}$ (U open) satisfying:

(1) $L(x_*) = 0$

(2) $L(\vec{x}) > 0$ on $U \setminus \{\vec{x}_*\}$

(3) $\frac{d}{dt} L(\underbrace{\vec{x}(t)}_{\text{solution to } \vec{x}' = \vec{F}(\vec{x})}) \leq 0$ whenever $\vec{x}(t) \neq \vec{x}_*$.

$\Longrightarrow$ $\vec{x}_*$ is stable.

(3)' $\frac{d}{dt} L(\vec{x}(t)) < 0$ whenever $\vec{x}(t) \neq \vec{x}_*$ $\Longrightarrow$ $\vec{x}_*$ is asymptotically sts.