1 a) $A = \left[\begin{array}{c|c} Q & I \\ \hline 0 & Q \end{array}\right]$, $A^2 = \left[\begin{array}{c|c} Q & I \\ \hline 0 & Q \end{array}\right]\left[\begin{array}{c|c} Q & I \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} Q^2 & 2Q \\ \hline 0 & Q^2 \end{array}\right]$,

$$A^3 = \left[\begin{array}{c|c} Q^2 & 2Q \\ \hline 0 & Q^2 \end{array}\right]\left[\begin{array}{c|c} Q & I \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} Q^3 & 3Q^2 \\ \hline 0 & Q^3 \end{array}\right].$$

<u>Guess</u>: $A^n = \left[\begin{array}{c|c} Q^n & nQ^{n-1} \\ \hline 0 & Q^n \end{array}\right]$ $\forall n = 0, 1, 2, \ldots$

It can be proved by induction: Assume $A^k = \left[\begin{array}{c|c} Q^k & kQ^{k-1} \\ \hline 0 & Q^k \end{array}\right]$ $\exists k$,

then $A^{k+1} = \left[\begin{array}{c|c} Q^k & kQ^{k-1} \\ \hline 0 & Q^k \end{array}\right]\left[\begin{array}{c|c} Q & I \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} Q^{k+1} & Q^k + kQ^k \\ \hline 0 & Q^{k+1} \end{array}\right] = \left[\begin{array}{c|c} Q^{k+1} & (k+1)Q^k \\ \hline 0 & Q^{k+1} \end{array}\right]$.

$$\therefore e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{t^n Q^n}{n!} & \frac{t^n Q^{n-1} n}{n!} \\ 0 & \frac{t^n Q^n}{n!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{tQ} & t + \frac{t^2 Q}{1!} + \frac{t^3 Q^2}{2!} + \frac{t^4 Q^3}{3!} + \cdots \\ 0 & e^{tQ} \end{bmatrix}$$

$$= \begin{bmatrix} e^{tQ} & te^{tQ} \\ 0 & e^{tQ} \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos\beta t & \sin\beta t & t\cos\beta t & t\sin\beta t \\ -\sin\beta t & \cos\beta t & -t\sin\beta t & t\cos\beta t \\ 0 & 0 & \cos\beta t & \sin\beta t \\ 0 & 0 & -\sin\beta t & \cos\beta t \end{bmatrix} \#$$

Recall from HW1: $e^{tQ} = e^{\alpha t} \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}$

b) Solution to the IVP: $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ is given by

$\varphi_t(\vec{x}_0) = e^{tA}\vec{x}_0$ where entries of e^{tA} are of $O(te^{\alpha t})$-order

$\therefore \forall \vec{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ <u>unit</u>, we have

$$\left|e^{tA}\vec{u}\right| \le \left(\sum_{i=1}^{4}\Big(\sum_{j=1}^{4} \underbrace{[e^{tA}]_{ij}}_{\le O(te^{\alpha t})} u_j\Big)^2\right)^{1/2} \le \left(\sum_{i=1}^{4} O(te^{\alpha t})^2 \underbrace{\sum_{j=1}^{4} u_j^2}_{=1}\right)^{1/2} \le O(te^{\alpha t})$$

$$\therefore \|e^{tA}\| \le O(te^{\alpha t}) \Rightarrow |\varphi_t(\vec{x}_0)| \le \|e^{tA}\|\,|\vec{x}_0| \le O(te^{\alpha t})\,|\vec{x}_0|.$$

If $\alpha < 0$, $te^{\alpha t} \to 0$ as $t \to +\infty$. In particular, $\exists C > 0$ s.t.

$te^{\alpha t} \le C$ $\forall t \ge 0$.

$\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{C}$ s.t. whenever $\vec{x}_0 \in B_\delta(\vec{0})$, $|\varphi_t(\vec{x}_0)| \le C|\vec{x}_0| < C\delta = \varepsilon$.

Also, $\lim_{t\to\infty} |\varphi_t(x_0)| = 0$ as $te^{\alpha t} \to 0$. $\therefore \vec{0}$ is asymptotically stable.

2(a) For each $\varepsilon > 0$, we define

$$v_\varepsilon(t) := G^{-1}\left(G(c+\varepsilon) + \int_0^t p(s)\,ds\right) \rightsquigarrow v_\varepsilon(0) = G^{-1}(G(c+\varepsilon)) = c+\varepsilon.$$

then $G(v_\varepsilon(t)) = G(c+\varepsilon) + \int_0^t p(s)\,ds$

$\overset{\frac{d}{dt}}{\Rightarrow} \quad G'(v_\varepsilon(t))\, v_\varepsilon'(t) = p(t)$

$\Rightarrow \quad \dfrac{1}{\psi(v_\varepsilon(t))} \cdot v_\varepsilon'(t) = p(t) \longleftarrow G'(r) = \dfrac{d}{dr}\displaystyle\int_0^r \frac{1}{\psi(s)}\,ds = \frac{1}{\psi(r)}$

$\Rightarrow \quad v_\varepsilon'(t) = p(t)\,\psi(v_\varepsilon(t))$

$\overset{\text{from class}}{\Rightarrow} \quad v_\varepsilon(t) = v_\varepsilon(0) + \int_0^t p(s)\,\psi(v_\varepsilon(s))\,ds$

$$= c+\varepsilon + \int_0^t p(s)\,\psi(v_\varepsilon(s))\,ds. \longrightarrow (\dagger)$$

We claim that $u(t) < v_\varepsilon(t) \quad \forall t \geq 0$.

<u>Proof of claim:</u> suppose not, and note that $u(0) \leq c + \cancel{\int_0^0} < c+\varepsilon = v_\varepsilon(0)$.

then $\exists$ the first time $t_1 > 0$ s.t. (ψ strictly $\nearrow$)

$$\begin{cases} u(t) < v_\varepsilon(t) & \text{on } t \in [0, t_1) \Rightarrow \psi(u(t)) < \psi(v_\varepsilon(t)) \text{ on } [0,t_1) \\ u(t_1) = v_\varepsilon(t_1) \Rightarrow \psi(u(t_1)) = \psi(v_\varepsilon(t_1)). \end{cases}$$

then $\displaystyle\int_0^{t_1} \underbrace{p(s)}_{\oplus} \underbrace{\left(\psi(v_\varepsilon(s)) - \psi(u(s))\right)}_{\oplus} ds > 0$

However, we also have:

$$\int_0^{t_1} p(s)\,\psi(v_\varepsilon(t)) - p(s)\,\psi(u(s))\,ds$$

$$\leq \underbrace{\left(v_\varepsilon(t_1) - (c+\varepsilon)\right)}_{\text{from } (\dagger)} - \underbrace{\left(u(t_1) - (c+\varepsilon)\right)}_{\text{given}} = v_\varepsilon(t_1) - u(t_1) = 0.$$

$\therefore$ Contradiction! $\blacksquare$

$$\therefore \quad u(t) < v_\varepsilon(t) = G^{-1}\left(G(c+\varepsilon) + \int_0^t p(s)\,ds\right) \quad \forall t \geq 0$$

$\displaystyle\int_0^r \frac{1}{\psi(s)}\,ds$ integrable $\Rightarrow G(r)$ is continuous $\Rightarrow G^{-1}$ continuous $\Bigg\} \Rightarrow \displaystyle\lim_{\varepsilon \to 0} v_\varepsilon(t) = G^{-1}\left(G(c) + \int_0^t p(s)\,ds\right)$

$$\Rightarrow u(t) \leq G^{-1}\left(G(c) + \int_0^t p(s)\,ds\right) \quad \forall t \geq 0.$$

(b) $\vec{x}(t)$, $\vec{y}(t)$ satisfy:

$$\vec{x}(t) = \vec{p} + \int_0^t \vec{F}(\vec{x}(s))\,ds, \qquad \vec{y}(t) = \vec{p} + \int_0^t \vec{F}(\vec{y}(s))\,ds$$

$$\Rightarrow \underbrace{|\vec{x}(t) - \vec{y}(t)|}_{u(t)} = \left| \int_0^t \vec{F}(\vec{x}(s)) - \vec{F}(\vec{y}(s))\,ds \right|$$

$$\leq \int_0^t |\vec{F}(\vec{x}(s)) - \vec{F}(\vec{y}(s))|\,ds$$

$$\leq \int_0^t \underbrace{e^{-s}}_{\substack{p(s) \\ +}} \underbrace{\sqrt{|\vec{x}(s) - \vec{y}(s)|}}_{\psi \circ u(s)}\,ds \quad \text{where } \psi(s) := \sqrt{s}.$$

$$G(r) = \int_0^r \frac{1}{\sqrt{s}}\,ds = \underline{\underline{2\sqrt{r}}} < \infty.$$

$$\Rightarrow G^{-1}(r) = \left(\frac{r}{2}\right)^2.$$

From (a), we have

$$u(t) := |\vec{x}(t) - \vec{y}(t)| \leq G^{-1}\Big(\underbrace{G(0)}_{=0} + \int_0^t e^{-s}\,ds\Big)$$

$$= \left(\frac{\int_0^t e^{-s}\,ds}{2}\right)^2 = \frac{(1-e^{-t})^2}{4} \quad \divideontimes.$$

3(a) When $t \in [0, \frac{\delta}{n}]$,

$$x_0 + \int_0^t F(y_n(s))\,ds = x_0 + \int_0^t \underbrace{F(x_0)}_{\text{constant}}\,ds = x_0 + t F(x_0) = x_n(t).$$

$s \in [0, \frac{\delta}{n}) \Rightarrow y_n(s) = x_0$

Now suppose $t \in \left(\frac{j\delta}{n}, \frac{(j+1)\delta}{n}\right]$ $\exists\, j = 1, 2, \ldots, n-1$.

then $x_0 + \int_0^t F(y_n(s))\,ds$

$$= x_0 + \int_0^{\delta/n} F(y_n(s))\,ds + \int_{\delta/n}^{2\delta/n} F(y_n(s))\,ds + \cdots + \int_{j\delta/n}^{t} F(y_n(s))\,ds$$

$$= x_0 + \int_0^{\delta/n} F(x_0)\,ds + \int_{\delta/n}^{2\delta/n} F\left(x_n\left(\tfrac{\delta}{n}\right)\right)ds + \cdots + \int_{j\delta/n}^{t} F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)ds$$

$$= x_0 + F(x_0)\cdot\left(\frac{\delta}{n} - 0\right)$$
$$+ F\left(x_n\left(\tfrac{\delta}{n}\right)\right)\cdot\left(\frac{2\delta}{n} - \frac{\delta}{n}\right)$$
$$+ \cdots$$
$$+ F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)\left(t - \frac{j\delta}{n}\right)$$

$$= x_0 + \frac{\delta}{n}\left(F(x_0) + F\left(x_n\left(\tfrac{\delta}{n}\right)\right) + \cdots + F\left(x_n\left(\tfrac{(j-1)\delta}{n}\right)\right)\right)$$
$$+ \left(t - \frac{j\delta}{n}\right) F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)$$

Now consider

$$x_n\left(\frac{(k+1)\delta}{n}\right) = x_n\left(\frac{k\delta}{n}\right) + \left(\frac{(k+1)\delta}{n} - \frac{k\delta}{n}\right) F\left(x_n\left(\tfrac{k\delta}{n}\right)\right) \quad \forall k = 1, 2, \ldots, n-1.$$

$$= x_n\left(\frac{k\delta}{n}\right) + \frac{\delta}{n} F\left(x_n\left(\tfrac{k\delta}{n}\right)\right)$$

$$\Rightarrow \frac{\delta}{n} F\left(x_n\left(\tfrac{k\delta}{n}\right)\right) = x_n\left(\frac{(k+1)\delta}{n}\right) - x_n\left(\frac{k\delta}{n}\right)$$

$$\Rightarrow \frac{\delta}{n}\left(F(x_0) + F\left(x_n\left(\tfrac{\delta}{n}\right)\right) + \cdots + F\left(x_n\left(\tfrac{(j-1)\delta}{n}\right)\right)\right)$$

$$= \frac{\delta}{n} F(x_0) + \sum_{k=1}^{j-1}\left(x_n\left(\tfrac{(k+1)\delta}{n}\right) - x_n\left(\tfrac{k\delta}{n}\right)\right)$$

$$= \left(x_n\left(\tfrac{\delta}{n}\right) - x_0\right) + \underbrace{x_n\left(\tfrac{j\delta}{n}\right) - x_n\left(\tfrac{\delta}{n}\right)}_{\text{telescoping sum}} = -x_0 + x_n\left(\tfrac{j\delta}{n}\right).$$

def. of $x_n(\frac{\delta}{n})$

$$\therefore\ x_0 + \int_0^t F(y_n(s))\,ds = x_0 + \left(-x_0 + x_n\left(\tfrac{j\delta}{n}\right)\right) + \left(t - \tfrac{j\delta}{n}\right) F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)$$

$$= x_n\left(\tfrac{j\delta}{n}\right) + \left(t - \tfrac{j\delta}{n}\right) F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)$$

$$= x_n(t) \quad \text{as desired.}$$

(b) First show that $\{x_n(t)\}$ is uniformly bounded on $[0,\delta]$

By (a): $\quad x_n(t) = x_0 + \int_0^t F(y_n(s))\,ds \quad \forall n \in \mathbb{N}.$

We argue that $y_n(s) \in \overline{B_r(x_0)} \quad \forall s \in [0,\delta]$.

It suffices to show $x_n\left(\frac{j\delta}{n}\right) \in \overline{B_r(x_0)} \quad \forall j = 1, 2, \dots, n-1$.

When $j=1$, $\quad x_n\left(\frac{\delta}{n}\right) = x_0 + \frac{\delta}{n} F(x_0)$

$$\Rightarrow \left|x_n\left(\tfrac{\delta}{n}\right) - x_0\right| = \tfrac{\delta}{n} |F(\underbrace{x_0}_{\in \overline{B_r(x_0)}})| \le \tfrac{\delta}{n} M \underset{\text{by } \delta < \frac{r}{M}.}{<} \tfrac{r}{n}$$

$$\therefore\ x_n\left(\tfrac{r}{n}\right) \in B_{\frac{r}{n}}(x_0) \subset \overline{B_r(x_0)}.$$

When $j=2$, $\quad x_n\left(\frac{2\delta}{n}\right) = x_n\left(\frac{\delta}{n}\right) + \left(\frac{2\delta}{n} - \frac{\delta}{n}\right) F\left(x_n\left(\frac{\delta}{n}\right)\right)$

$$\Rightarrow \left|x_n\left(\tfrac{2\delta}{n}\right) - x_n\left(\tfrac{\delta}{n}\right)\right| = \tfrac{\delta}{n} \left|F(\underbrace{x_n\left(\tfrac{\delta}{n}\right)}_{\in \overline{B_r(x_0)}})\right| \le \tfrac{\delta M}{n} < \tfrac{r}{n}$$

$$\therefore\ \left|x_n\left(\tfrac{2\delta}{n}\right) - x_0\right| \le \left|x_n\left(\tfrac{2\delta}{n}\right) - x_n\left(\tfrac{\delta}{n}\right)\right| + \left|x_n\left(\tfrac{\delta}{n}\right) - x_0\right| < \tfrac{r}{n} + \tfrac{r}{n} = \tfrac{2r}{n}.$$

$$\Rightarrow x_n\left(\tfrac{2\delta}{n}\right) \in B_{\frac{2r}{n}}(x_0) \subset \overline{B_r(x_0)}.$$

Generally, we have $\forall j = 1, 2, \dots, n-1$

$$\left|x_n\left(\tfrac{(j+1)\delta}{n}\right) - x_n\left(\tfrac{j\delta}{n}\right)\right| = \tfrac{\delta}{n}\left|F\left(x_n\left(\tfrac{j\delta}{n}\right)\right)\right|$$

Using this, one can prove by induction on j that $\left|x_n\left(\frac{j\delta}{n}\right) - x_0\right| < \frac{jr}{n}$. $\forall j = 1, 2, \dots, n-1.$

$$\therefore\ x_n\left(\tfrac{j\delta}{n}\right) \in B_{\frac{jr}{n}}(x_0) \subset \overline{B_r(x_0)}.$$

By definition of $y_n(t) = \begin{cases} x_0 & \text{if } t \in [0, \frac{\delta}{n}] \\ x_n\left(\frac{j\delta}{n}\right) & \text{if } t \in \left(\frac{j\delta}{n}, \frac{(j+1)\delta}{n}\right] \end{cases}, \ j = 1, 2, \dots, n-1.$

we conclude $y_n(t) \in \overline{B_r(x_0)} \quad \forall n \ge 1,\ t \in [0,\delta]$

$$\text{(a)} \Rightarrow |x_n(t) - x_0| = \left|\int_0^t F(\underbrace{y_n(s)}_{\in \overline{B_r(x_0)}})\,ds\right| \le \int_0^t \underbrace{|F(y_n(s))|}_{\le M}\,ds \le Mt \le M\delta. \quad \forall t \in [0,\delta].$$

$$\therefore\ |x_n(t)| \le |x_0| + M\delta \quad \forall t \in [0,\delta],\ n \ge 1.$$

$\therefore \{x_n(t)\}$ is uniformly bounded on $[0,\delta]$.

For equicontinuity, we consider any $t, \tau \in [0, \delta]$ with $t < \tau$:

$$|x_n(t) - x_n(\tau)| = \left| x_0 + \int_0^t F(y_n(s))\,ds - x_0 - \int_0^\tau F(y_n(s))\,ds \right|$$

$$= \left| \int_t^\tau F(y_n(s))\,ds \right| \le \int_t^\tau |F(\underbrace{y_n(s)}_{\in \overline{B_r(x_0)}})|\,ds \le M(\tau - t)$$

$\therefore\ \forall \varepsilon > 0,\ \exists \delta := \frac{\varepsilon}{M+1}$ s.t. $\forall t, \tau \in [0, \delta]$ s.t. $|t - \tau| < \delta$,

we have $|x_n(t) - x_n(\tau)| < M|\tau - t| < M\delta \underset{\delta = \frac{\varepsilon}{M+1}}{<} \varepsilon \quad \forall n \ge 1.$

$\therefore\ \{x_n(t)\}$ is equicontinuous on $[0, \delta]$.

By Arzela-Ascoli's Theorem, $\exists$ subsequence $\{x_{n_i}(t)\}_{i=1}^\infty$ converging uniformly on $[0, \delta]$ to a limit function $x_\infty(t) : [0, \delta] \to \mathbb{R}$.