- Coverage : Lecture 08 ~ 09
  - Uniqueness Theorem for ODE Systems. $\vec{x}' = \vec{F}(\vec{x}, t)$ OR $\vec{x}' = \vec{G}(\vec{x})$

① Grönwall's Inequality. (☆) Need $\vec{F}, \vec{G}$ Lipschitz. (or locally)

② Continuous Dependence Inequality for Nonlinear Systems. In chap. 01. $|\vec{x}(t) - \vec{y}(t)| \leq |\vec{x}(0) - \vec{y}(0)| e^{\|A\| |t|}$

③ Uniqueness Theorem

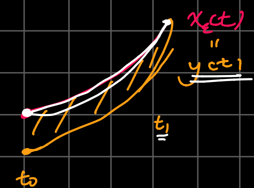
⇒ ④ Flow Map of Autonomous System.

- Gronwall's Inequality $y : I \to \mathbb{R}$, continuous.

Let $y \in C(I)$ satisfies $y(t) \leq C + \int_{t_0}^{t} L y(s)\, ds$ for some $C \in \mathbb{R}$, $L > 0$,

$< (C + \varepsilon)$ $\quad$ $x(s)$

then $y(t) \leq Ce^{L|t-t_0|}$.



- Idea of the Proof : (Barrier Method)

① Construct "Barrier" functions $x_\varepsilon(t)$ satisfies the integral equations.

② Show that $y(t) < x_\varepsilon(t)$ $\forall\, \varepsilon > 0$ on I. ($x(t)$)

③ Let $\varepsilon \to 0^+$, then $y(t) \leq x(t) = Ce^{L|t-t_0|}$

Lecture : $t > t_0$ $\quad \boxed{t < t_0}$

- Another proof : (Not using the barrier method) $\quad$ Case $t > t_0$. $(L > 0)$

$$y(t) \leq C + \int_{t_0}^{t} L y(s)\, ds. \quad (= t_0.)$$

Define $x(t) = C + \int_{t_0}^{t} L y(s)\, ds$.

Notice that $\boxed{x'(t) = L y(t)}$.

$\forall\, t > t_0$ $\Rightarrow$ $L y(t) \leq L x(t)$

$\Rightarrow$ $\underbrace{L y(t) - x'(t)}_{=0} \leq L x(t) - x'(t)$

$\Rightarrow$ $x'(t) - L x(t) \leq 0$ (☆)

Multiply $e^{-L(t-t_0)}$ on (☆) : $\underbrace{e^{-L(t-t_0)} x'(t) - Le^{-L(t-t_0)} x(t)}_{[e^{-L(t-t_0)} x(t)]'} \leq 0$. ↙ Product Rule.

$\Rightarrow$ $[e^{-L(t-t_0)} x(t)]' \leq 0$

integrat from t_0 to t.

$\Rightarrow$ $e^{-L(t-t_0)} x(t) - \underbrace{e^{-L(t_0-t_0)}}_{=1} \underbrace{x(t_0)}_{=C} \leq 0$

$\Rightarrow$ $x(t) \leq Ce^{L(t-t_0)}$

$\therefore$ $y(t) \leq x(t) \leq Ce^{L(t-t_0)}$ #

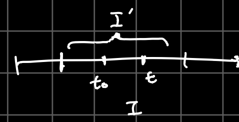
- Continuous Dependence Inequality for Nonlinear Systems.

Let $\Omega \subset \mathbb{R}^d$ be an open domain, and I be a time interval.

Suppose $\vec{F} : \Omega \times I \to \mathbb{R}^d$ is a Lipschitz continuous vector field on $\Omega \times I$.

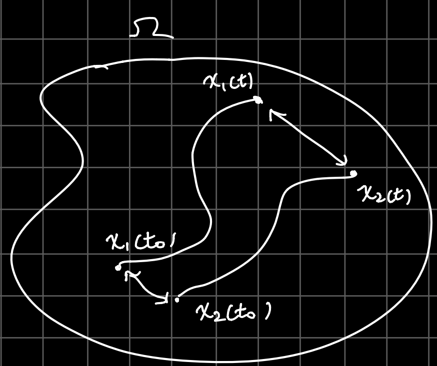
If $\vec{x}_1(t)$, $\vec{x}_2(t)$ are both solutions to $\vec{x}' = \vec{F}(\vec{x}, t)$, and

$\vec{x}_1(t)$, $\vec{x}_2(t) \in \Omega$, for some $t \in I' \subset I$,



then we have $|\vec{x}_1(t) - \vec{x}_2(t)| \leq |\vec{x}_1(t_0) - \vec{x}_2(t_0)| \underbrace{e^{L|t-t_0|}}_{\text{from Grönwall's Ineq.}}$ — constant.

for any $t, t_0 \in I'$.

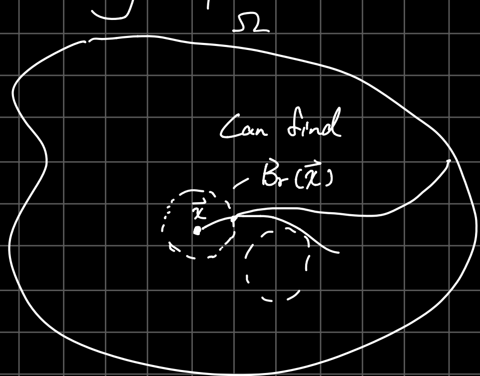


- Cor : Uniqueness Thm for Lipschitz.

Apply Grönwall's Inequality on

$$|\vec{x}_1(t) - \vec{x}_2(t)| \leq |x_1(t_0) - x_2(t_0)| + \int |F \underline{\qquad\quad}| \, ds$$

- For locally Lipschitz case.

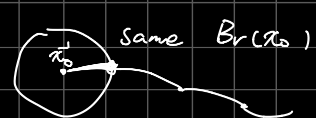


s.t. $\vec{F}$ is Lipschitz cont. on $B_r(\vec{x}) \times I$.

Lipschitz constant $L(\vec{x})$

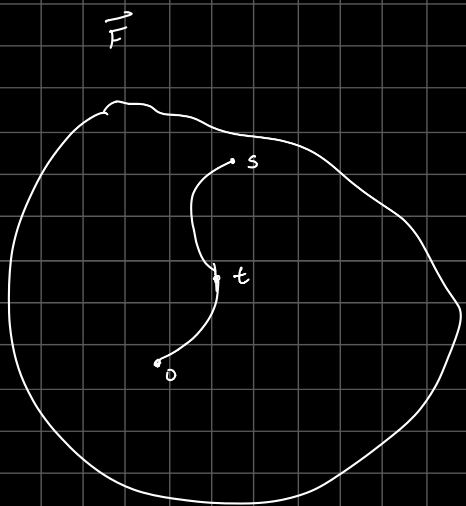
$\Rightarrow$ Difficulty on continuous dependence.

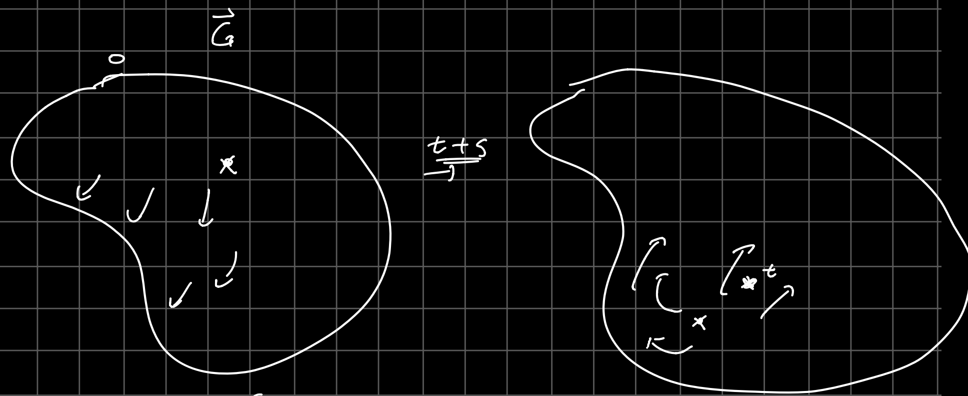
Idea for Uniqueness Thm : $\vec{x}_1(t_0) = \vec{x}_2(t_0)$
$= \vec{x}_0$

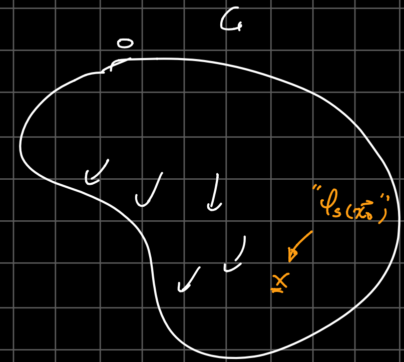


• Flow map for Autonomous system. $\vec{F}(\vec{x})$ $\boxed{\varphi_t(\varphi_s(\vec{x}_0)) = \varphi_{t+s}(\vec{x}_0)}$

$\boxed{\text{Non-Autonomous system } \vec{G}(\vec{x}, t)}$ why cannot $\varphi_t(x)$





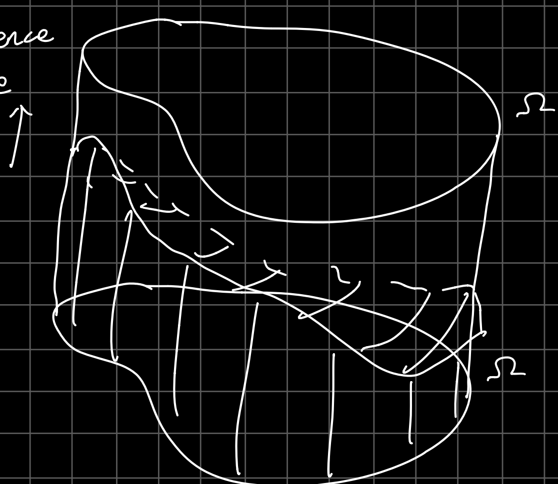


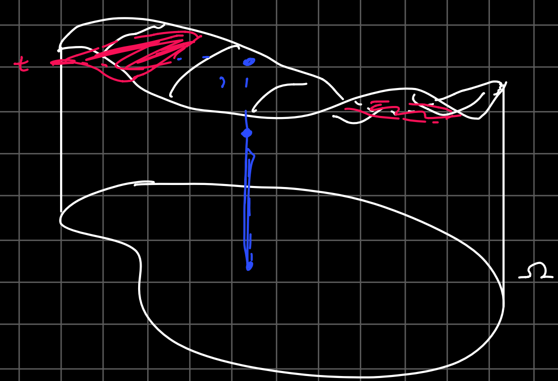
Domain for $\varphi(\vec{x}, t)$ $\boxed{\text{Focus on } \varphi_t(\vec{x})}$

maximal existence time interval

$$\Sigma(\vec{F}) := \bigcup_{x_0 \in \Omega} \{\vec{x}_0\} \times I_{x_0}$$

existence time





Fixed t,

$\varphi_t(\vec{x}) : \Sigma_t \to$

$\boxed{\text{cont.}}$ $\boxed{\text{Open set}}$