- Chapter 02 : Existence and Uniquess.

$$\begin{cases} \vec{x}' = \vec{F}(\vec{x}, t) \\ \vec{x}(0) = \vec{x}_0 \end{cases} \quad \text{OR} \quad \begin{cases} \vec{x}' = \vec{G}(\vec{x}) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

Non-Autonomous Autonomous.

- Prop. 2.3 : Let $\vec{F}$ be continuous, $\vec{x}_0 \in \mathbb{R}^d$

$\vec{x}(t)$ solves $\begin{cases} \vec{x}' = \vec{F}(\vec{x}, t) \\ \vec{x}(0) = \vec{x}_0 \end{cases} \Leftrightarrow$ $\vec{x}(t)$ is a continuous solution to $\vec{x}(t) = \vec{x}_0 + \int_0^t \vec{F}(\vec{x}(s), s)\, ds$

Idea :

" $\vec{x}_n(t) = \vec{x}_0 + \int_0^t \vec{F}(\vec{x}_n(s), s)\, ds$ "

$$\lim_{n\to\infty} \vec{x}_n(t) = \vec{x}_0 + \lim_{n\to\infty} \int_0^t \vec{F}(\vec{x}_n(s), s)\, ds$$

- Picard-Lindelöf Existence Theorem.
  - Picard iteration sequence. $\{\vec{x}_n(t)\}_{n=0}^{\infty}$ for $\begin{cases} \vec{x}'(t) = \vec{F}(\vec{x}, t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$

$$\vec{x}_0(t) = \vec{x}_0 \quad (\text{Abuse of notation})$$

$$\vec{x}_n(t) := \vec{x}_0 + \int_0^t \vec{F}(\vec{x}_{n-1}(s), s)\, ds$$

- Example : Consider $\begin{cases} x'' = -x \\ x(0) = 0, \quad x'(0) = 1. \end{cases}$

Let $v = x'$, then

$$\vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}, \quad \vec{x}' = \begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} v \\ -x \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By prop. 2.3. The IVP is equivalent to

$$\vec{x}(t) = \vec{x}_0 + \int_0^t \vec{F}(\vec{x}(s))\, ds.$$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} v(s) \\ -x(s) \end{bmatrix} ds, \quad \vec{x}_0(t) = \vec{x}_0$$

$$\vec{x}_1(t) = \begin{bmatrix} 0 + \int_0^t v_0(s)\, ds \\ 1 - \int_0^t x_0(s)\, ds \end{bmatrix} = \begin{bmatrix} \int_0^t 1\, ds \\ 1 - \int_0^t 0\, ds \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\vec{x}_2(t) = \begin{bmatrix} t \\ 1 - t^2/2 \end{bmatrix}, \quad \vec{x}_3(t) = \begin{bmatrix} t - t^3/3! \\ 1 - t^2/2 \end{bmatrix}.$$

$$\vec{x}_4(t) = \begin{bmatrix} t - t^3/3! \\ 1 - t^2/2 + t^4/4! \end{bmatrix}$$

"Guess":

$$\vec{x}_{2n}(t) = \begin{bmatrix} \sum_{k=1}^{n} \frac{(-1)^{k-1} t^{2k-1}}{(2k-1)!} \\ \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{(2k)!} \end{bmatrix} \qquad n \in \mathbb{N}.$$

$$\vec{x}_{2n-1}(t) = \begin{bmatrix} \sum_{k=1}^{n} \frac{(-1)^{k-1} t^{2k-1}}{(2k-1)!} \\ \sum_{k=0}^{n-1} \frac{(-1)^k t^{2k}}{(2k)!} \end{bmatrix}$$

Can prove by induction.

As $n \to \infty$, $\sum_{k=1}^{n} \frac{(-1)^{k-1} t^{2k-1}}{(2k-1)!} \to \sin t$

$\sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{(2k)!} \to \cos t.$

$$\Rightarrow \vec{x}_\infty(t) = \begin{bmatrix} x_\infty(t) \\ v_\infty(t) \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\begin{cases} x'' = -x \\ \underline{\underline{x(0) = 0}}, \ \underline{\underline{x'(0) = 1}} \end{cases}$$

$\Downarrow$ $A = 0$ $\qquad$ $\Downarrow$ $B = 1$

Plug $\underline{\underline{e^{rt}}}$ into it. $\Rightarrow r = \pm i$.

General Sol: $x(t) = A\cos t + B\sin t.$

$\Rightarrow x(t) = \sin t.$

- <u>Lipschitz Continuity</u> / Locally Lipschitz continuous.

  $\vec{F} : \Omega \times I \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^d$

  $\vec{F}$ is Lipschitz cont. $\overset{\text{def}}{\Longleftrightarrow}$ $\exists L > 0$, s.t.

  $|\vec{F}(\vec{x}, t) - \vec{F}(\vec{y}, t)| \leq L |\vec{x} - \vec{y}|$ $\forall \vec{x}, \vec{y} \in \Omega$.

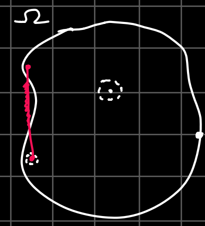
  $\Rightarrow$ $\vec{F}$ is cont. on $\Omega \times I$. Pick $\delta = \frac{\varepsilon}{L}$.

- Lipschitz cont. func.: $\sin(x)$, $\cos(x)$ (on $\mathbb{R}$)
- Lipschitz cont. on $\mathbb{R}$, but not differentiable everywhere: $|x|$
- Lipshitz cont. on $\mathbb{R}$, differentiable on $\mathbb{R}$, but not $C^1$: $\underline{\underline{f}}(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
- Not Lipshitz cont. on $\mathbb{R}$, but are $C^\infty$: $\underline{\underline{x^2}}$, $\underline{\underline{e^x}}$

  (But locally Lipschitz cont. on $\mathbb{R}$)

- Locally Lipschitz continuous on $\Omega \times I$. ← Open.

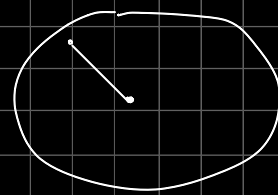
$\vec{F} : \Omega \times I \to \mathbb{R}^m$.





if $\forall \vec{x}_0 \in \Omega$, $\exists$ open ball $B_r(\vec{x}_0) \subset \Omega$, s.t.

$\vec{F}$ is Lipschitz cont. on $B_r(\vec{x}_0) \times I$.



- If $\vec{F}$ is differentiable on $\Omega$ and $I$ ← convex.

we have $|x|$ is not diff. at $x=0$.

$\vec{F}$ is Lipschitz cont. on $\Omega \times I$

$\Leftrightarrow$ $\left|\frac{\partial F_i}{\partial x_j}\right| \leq C$ for all $i, j$, for all $(\vec{x}, t) \in \Omega \times I$.

$\Leftrightarrow$ $\|\underbrace{D\vec{F}}_{\text{Jacobian}}\| \leq \tilde{C}$ $\forall (\vec{x}, t) \in \Omega \times I$.

- $C^1 \Rightarrow$ locally Lipshitz cont. (on $\Omega \times I$ ) ← open. convex.

↑ closed, bounded.

- Examples.

Show $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ . is Lipshitz cont. on $\mathbb{R}$.
differentable on $\mathbb{R}$.
but not $C^1$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0 \Rightarrow f$ is cont. on $\mathbb{R}$.

For $x \neq 0$, $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.

For $x = 0$, $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

but $\lim_{x\to 0} 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) = \lim_{x\to 0} -\cos(\frac{1}{x}) \neq 0$

$f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.

$|f'(x)| = 2|x\sin(\frac{1}{x})| + \underbrace{|\cos(\frac{1}{x})|}_{\leq 1} \leq 3. \Rightarrow f$ is Lip. cont.

$$\lim_{x \to \infty} \frac{\sin(\ldots)}{1/x} = 1$$

- Composition of Lipschitz cont. functions.

  $F : \Omega \to \mathbb{R}$ , $G : F(\Omega) \to \mathbb{R}$

  $G \circ F : \Omega \to \mathbb{R}$ is Lipshitz continuous on $\Omega$

  For $x, y \in \Omega$ ,

  $|G(F(x)) - G(F(y))|$ (Since $F(x), F(y) \in F(\Omega)$ )

  $\leq L_G \, |F(x) - F(y)|$

  $\leq L_F \cdot L_G \cdot |x - y| = L|x - y|$ . $\square$