- Today's Coverage : Chapter 01 (linear System of ODE)
  - Review : Matrix Exponential , Matrix Norm ,
    Existence and uniqueness of solution of linear system $\vec{x}' = A\vec{x}$
  - Examples on calculating matrix exponentials for different A.

In 1D : $x' = ax$, $x(0) = x_0$ $\Rightarrow$ $x(t) = e^{at} x_0$

$\Rightarrow$ Generalize : $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x_0}$ $\overset{?}{\Rightarrow}$ $x(t) = e^{At}\vec{x_0}$

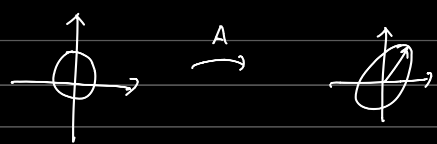
Two Problems : ① $e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$ $\to$ A^k why it converges ?

② show $\frac{d}{dt} e^{At} = A e^{At}$.

Need matrix norm for that :

$\|A\| := \sup \{ |A\vec{x}| : |\vec{x}| = 1 \}$

CAN BE NON-SQUARE



Five properties of it : ① $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = 0$

② $\|cA\| = |c| \cdot \|A\|$ $\forall c \in \mathbb{R}$

③ $\|A + B\| \leq \|A\| + \|B\|$

④ $|A\vec{x}| \leq \|A\| \cdot |\vec{x}|$ $\forall \vec{x} \in \mathbb{R}^d$.

⑤ $\|AB\| \leq \|A\| \cdot \|B\|$ (sub-multiplicative)

$\Rightarrow$ $\|A^n\| \leq \|A\|^n$ for square A.

$\Rightarrow$ Show $\vec{x}(t) = e^{At}\vec{x_0}$ is a solution to $\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x_0} \end{cases}$.

- Examples

① $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $e^D = \begin{bmatrix} e^{\lambda_1} & \\ & e^{\lambda_2} \end{bmatrix}$

② $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $e^J = \begin{bmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{bmatrix}$

③ Consider e^{Jt}.

$J^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$, $J^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$, ... , $J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$

$$e^{Jt} = \sum_{k=0}^{\infty} \frac{J^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k} \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

Consider $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot k\lambda^{k-1} = \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot k\lambda^{k-1} = \sum_{k=1}^{\infty} \frac{t^k \lambda^{k-1}}{(k-1)!}$

$$= t \underbrace{\sum_{k=1}^{\infty} \frac{t^{k-1}\lambda^{k-1}}{(k-1)!}}_{e^{\lambda t}} = te^{\lambda t}$$

$Q = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. (Homework)

④ $A = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$. (Example 1.5)

observe the pattern, $A^2 = \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix} = -\theta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{bmatrix} = -\theta^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\dots$, $A^{2k} = (-1)^k \theta^{2k} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^{2k+1} = (-1)^k \theta^{2k+1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ wrong in the lecture notes

for $k \in \mathbb{N} \cup \{0\}$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!}$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}}_{\cos\theta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}}_{\sin\theta} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

⑤ Nilpotent matrix : if $A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some integer n.
then we say A is Nilpotent.

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \cdots + A^n .$$

$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow e^A = I + A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

- Exercise 1.20 : ① $e^{PAP^{-1}} = P\overbrace{e^A}^{e^A} P^{-1}$.

$$e^{PAP^{-1}} = \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PA^kP^{-1}}{k!} = P\left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right] P^{-1}$$

② $Ae^A = e^A \cdot A$

$$A \sum_k \frac{A^k}{k!} = \sum_k \frac{A^{k+1}}{k!} = \sum_k \frac{A^k \cdot A}{k!} = e^A \cdot A .$$

③ if $AB = BA$, then $e^A \cdot e^B = e^B \cdot e^A = \boxed{e^{A+B}}$.

Use it to express $\exp \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$.

$A = L + N$ — L: Diagonalizable, N: Nilpotent.

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} + \begin{bmatrix} & -\beta \\ \beta & \end{bmatrix}$$

- Continuous Dependence. (Thm. 1.29)

$A: d \times d$. $\vec{x}_0, \vec{y}_0 \in \mathbb{R}^d$

$\vec{x}(t), \vec{y}(t)$ solves $\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$, $\begin{cases} \vec{y}' = A\vec{y} \\ \vec{y}(0) = \vec{y}_0 \end{cases}$.

then $|\vec{x}(t) - \vec{y}(t)| \leq |\vec{x}_0 - \vec{y}_0| e^{\|A\| \cdot |t|}$ $\forall t \in \mathbb{R}$.

Recall Proof: $\frac{d}{dt} |\vec{x}(t) - \vec{y}(t)|^2 \dots$

Problematic Proof: we know $\boxed{\begin{aligned} \vec{x}(t) &= e^{At} \vec{x}_0 \\ \vec{y}(t) &= e^{At} \vec{y}_0 \end{aligned}}$.

$\underbrace{(\because \|e^{At}\| \leq e^{\|At\|})}$

$$|\vec{x}(t) - \vec{y}(t)| = |\underbrace{e^{At}}_{\text{matrix}} \underbrace{(\vec{x}_0 - \vec{y}_0)}_{\text{vector}}| \leq \|e^{At}\| \cdot |\vec{x}_0 - \vec{y}_0| \leq e^{\|A\| \cdot |t|} |\vec{x}_0 - \vec{y}_0|$$

We haven't show the solution is unique

① Generalize $x(t) = e^{at} x_0$ to $\underbrace{\vec{x}(t) = e^{At} \vec{x}_0}$. (Existence)

(use matrix norm) (i) why e^{At} is well-defined.
(ii) why $\frac{d}{dt} e^{At} = A e^{At}$.

② Continuous Dependence: $|\vec{x}(t) - \vec{y}(t)| \leq |\vec{x}_0 - \vec{y}_0| e^{\|A\| \cdot |t|}$ $\forall t \in \mathbb{R}$.

$\|A\|$ can be very large

if $|\vec{x}_0 - \vec{y}_0| \to 0 \Rightarrow$ Uniqueness of sol.

In Chapter 2: Nonlinear system. $\vec{x}' = \vec{F}(\vec{x}(t), t)$ / $\vec{G}(\vec{x}(t))$