Lecture 01 (20/02/2020)

System of ODE

e.g.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} x(t)^2 - y(t) \\ x(t) + y(t)^2 \end{bmatrix} = \underbrace{\begin{bmatrix} x^2 - y \\ x + y^2 \end{bmatrix}}_{\text{vector field}} \Bigg|_{(x(t),\, y(t))}$$

$\vec{F}(x,y) = (x^2 - y)\,\hat{i} + (x + y^2)\,\hat{j}$

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$.

$x(t)$, $y(t)$ unknown function.

Geometric meaning?

LHS : velocity vector of the solution curve.

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \leftarrow x(t)\,\hat{i} + y(t)\,\hat{j}$$

curve in $\mathbb{R}^2$.

solution to the ODE system.

$$\boxed{x''(t) + a x'(t) + b = 0}$$

$$\Leftrightarrow \begin{cases} x'(t) = y(t) \\ \underbrace{y'(t)}_{x''(t) = -ax'(t) - b} = -ax'(t) - b \end{cases}$$

Ch. 1.1 - 1.3.

linear system of ODE

$$\begin{cases} x'(t) = a\,x(t) + b\,y(t) \\ y'(t) = c\,x(t) + d\,y(t) \end{cases}$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\boxed{\vec{x}'(t) = A\,x(t)}$$

$$A = PDP^{-1}$$

$$\vec{x}'(t) = PDP^{-1}\vec{x}(t)$$

$$\Rightarrow \quad \underline{P^{-1}\vec{x}'(t)} = D\left(P^{-1}\vec{x}(t)\right)$$

$$\frac{d}{dt}\Big(\underbrace{P^{-1}x(t)}_{\vec{y}(t)}\Big) = D\,\underbrace{\left(P^{-1}x(t)\right)}_{\vec{y}(t)} \longrightarrow \frac{d}{dt}\vec{y}(t) = D\cdot\vec{y}(t)$$

easy to solve.

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\Leftrightarrow \quad \frac{d}{dt}y_1(t) = \lambda_1 y_1(t)$$

$$\frac{d}{dt}y_2(t) = \lambda_2 y_2(t).$$

$$\vec{x}(t) = P\vec{y}(t)$$

$$y_1(t) = C_1 e^{\lambda_1 t}$$

$$y_2(t) = C_2 e^{\lambda_2 t}$$

$\boxed{\text{Ch. 1.2}}$

Ch. 1.3

$$(*) \begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases} \qquad \vec{x}(t) \in \mathbb{R}^d.$$

(dim = 1) $\begin{cases} x'(t) = a x(t) \\ x(0) = x_0 \end{cases} \Longrightarrow x(t) = e^{at} x_0.$

$$= \left(1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots\right) x_0$$

Guess: $(*)$ solution:

$$\vec{x}(t) = \left(I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots\right)\vec{x}_0$$

Q1 Why converges?

$$[\ \cdot\ ] + [\ \cdot\ ] + [\ \cdot\ ] + [\ \cdot\ ] + \cdots$$

Q2: fine?

$$\vec{x}'(t) = \left(0 + A + 2t \cdot \frac{A^2}{2!} + 3t^2 \cdot \frac{A^3}{3!} + 4t^3 \cdot \frac{A^4}{4!} + \cdots\right)\vec{x}_0$$

$$= \left(A + \frac{tA^2}{1!} + \frac{t^2A^3}{2!} + \frac{t^3A^4}{3!} + \cdots\right)\vec{x}_0$$

$$= A\underbrace{\left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots\right)\vec{x}_0}_{\vec{x}(t)}$$

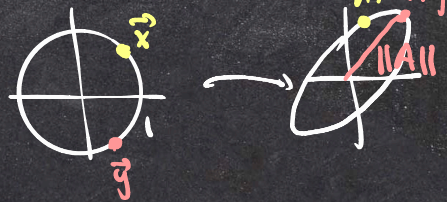
$$\Rightarrow \vec{x}\,'(t) = A\vec{x}(t).$$

$$\vec{x}(0) = \vec{x}_0$$

Norm of a matrix:

$$\|A\| := \sup\left\{ |A\vec{x}| : |\vec{x}| = 1 \right\}$$

$$\vec{x} \longmapsto A\vec{x}$$



① $\|cA\| = |c|\,\|A\|$

② $\|A\| \geq 0$, "$=$" iff $A = 0$

③ $\|A+B\| \leq \|A\| + \|B\|$.

④ $\|AB\| \le \|A\|\,\|B\|$ and $|A\vec{x}| \le \|A\|\,|\vec{x}|$.

Claim: $|A\vec{x}| \le \|A\|\,|\vec{x}|$.

Proof: if $\vec{x}=0 \leadsto$ trivial.

Assume $\vec{x} \neq 0$, $\vec{y} = \dfrac{\vec{x}}{\|\vec{x}\|}$ unit. $(|\vec{y}|=1)$

$$|A\vec{y}| \in \{|A\vec{z}| : |\vec{z}|=1\}$$

$$\Rightarrow |A\vec{y}| \le \sup\{|A\vec{z}| : |\vec{z}|=1\} =: \|A\|.$$

$$\left|A\left(\frac{\vec{x}}{|\vec{x}|}\right)\right| \le \|A\| \implies |A\vec{x}| \le \|A\|\,|\vec{x}|.$$

Claim: $\underbrace{\|AB\|}_{\substack{\shortparallel \\ \sup\{|AB\vec{x}| \,:\, |\vec{x}|=1\}}} \leq \|A\|\,\|B\|.$

Proof: WANT: $\|A\|\,\|B\|$ is an upper bound for $\{|AB\vec{x}| : |\vec{x}|=1\}$

$\forall |\vec{x}|=1$:

$$|A\underbrace{B\vec{x}}| \leq \|A\|\,|B\vec{x}| \leq \|A\|\,\|B\|\,|x| = \|A\|\,\|B\|.$$

$\overset{\sup}{\Longrightarrow}$ $\|AB\| \leq \|A\|\,\|B\|.$ ☒

Cor: $\|A^n\| \leq \|A\|^n.$

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