{ Normal version }

PROBLEM #1

(a) Theorem 3.15 (Liouville's Theorem)

(b) We need to show $f^{(2020)}(z) \equiv 0$:

For any $\alpha \in \mathbb{C}$, higher-order Cauchy's integral formula shows

$$f^{(2020)}(\alpha) = \frac{2020!}{2\pi i} \oint_{|z|=R} \frac{f(z)}{(z-\alpha)^{2021}}\, dz \quad \text{where } R > |\alpha|.$$

On $\{|z|=R\}$,

$$\left| \frac{f(z)}{(z-\alpha)^{2021}} \right| \le \frac{\sup_{|z|=R} |f(z)|}{\big||z|-|\alpha|\big|^{2021}}$$

$$\le \frac{\sup_{|z|=R} \frac{|f(z)|}{|p(z)|} \cdot \sup_{|z|=R} |p(z)|}{(R-|\alpha|)^{2021}}$$

let $p(z) = \sum_{j=0}^{2020} a_j z^{2020}$ $\longrightarrow$ $\le \dfrac{\sup_{|z|=R} \frac{|f(z)|}{|p(z)|} \cdot \sum_{j=0}^{2020} |a_j| R^{2020}}{(R-|\alpha|)^{2021}}$

$$\therefore \left| \oint_{|z|=R} \frac{f(z)}{(z-\alpha)^{2021}}\, dz \right| \le \underbrace{\sup_{|z|=R} \frac{|f(z)|}{|p(z)|}}_{\downarrow \text{ as } R\to\infty} \cdot \underbrace{\frac{\sum_{j=1}^{2020} |a_j| R^{2020} \cdot 2\pi R}{(R-|\alpha|)^{2021}}}_{\downarrow}$$

$$0 \cdot 2\pi \cdot a_{2020} = 0.$$

$$\therefore \left| f^{(2020)}(\alpha) \right| \le \frac{2020!}{2\pi} \left| \oint_{|z|=R} \frac{f(z)}{(z-\alpha)^{2021}}\, dz \right| \to 0 \text{ as } R \to \infty.$$

$\Longrightarrow f^{(2020)}(\alpha) = 0 \quad \forall \alpha \in \mathbb{C}.$

$\therefore$ f is a polynomial of degree ≤ 2019.

PROBLEM #2

(a) $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z} \leftarrow = 0 \iff z \in \mathbb{Z}$.

$\therefore$ Any integer $n \in \mathbb{Z}$ is a pole of $f(z)$.

Also 0 is a pole of $g \Rightarrow 0$ is also a pole of f.

To find residues:

$\forall n \in \mathbb{Z}, n \neq 0$, note that

$$\lim_{z \to n} (z-n) f(z) = \lim_{z \to n} \frac{\pi(z-n)\, g(z) \cdot \cos \pi z}{\sin \pi z}$$

$$= \lim_{z \to n} g(z) \cdot \frac{\overset{\to 0}{\pi z - \pi n}}{(-1)^n \sin(\pi z - \pi n)} \cos \pi z$$

$$= g(n) \cdot \frac{1}{(-1)^n} \cdot 1 \cdot \underbrace{\cos n\pi}_{=(-1)^n}$$

$$= g(n) \neq 0$$

$\therefore n \in \mathbb{Z} \setminus \{0\}$ is a simple pole of f and

$\operatorname{Res}(f, n) = g(n)$.

Now consider 0:

$$f(z) = g(z)\, \pi \cot \pi z = \frac{g(z)}{z} \pi z \cot \pi z$$

$$= \frac{g(z)}{z}\left(\pi i z + \sum_{n=0}^{\infty} \frac{B_n}{n!} (2\pi i z)^n\right) \quad \text{from HW4 Q2}$$

$$= \left(\sum_{m=-k}^{\infty} c_m z^m\right)\left(\pi i + \sum_{n=0}^{\infty} \frac{B_n}{n!} (2\pi i)^n z^{n-1}\right)$$

$$= \sum_{m=-k}^{\infty} c_m \pi i z^m + \sum_{\substack{n \geq 0 \\ m \geq -k}} \frac{B_n}{n!} c_m (2\pi i)^n z^{m+n-1}$$

We need to find the coefficient of $\frac{1}{z}$:

$$\sum_{m=-k}^{\infty} c_m \pi i z^m = c_{-k} \pi i z^{-k} + \cdots + \frac{c_{-1}\pi i}{z} + \cdots$$

For the second term, $z^{m+n-1} = \frac{1}{z} \iff m+n=0 \iff m=-n$.

$\therefore$ Coefficient of $\frac{1}{z}$ in the second sum

$$= \sum_{n \geq 0} \frac{B_n}{n!} c_{-n} (2\pi i)^n = \sum_{n=0}^{k} \frac{B_n}{n!} c_{-n} (2\pi i)^n.$$

$m=-n,\ m \geq -k \Rightarrow n \leq k$

$$\therefore \operatorname{Res}(f, 0) = \pi i c_{-1} + \sum_{n=0}^{k} \frac{B_n}{n!} c_{-n} (2\pi i)^n$$

(b) Consider the rectangular contour γ_N as in Example 4.11

Then γ_N encloses $0, \pm 1, \pm 2, \ldots, \pm N$ (which are poles of f).

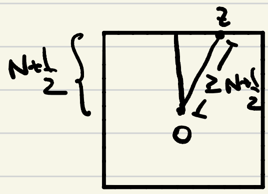
$$\Rightarrow \frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz = \sum_{n=-N}^{N} \operatorname{Res}(f,n)$$

$$= \begin{matrix} g(-N)+\cdots+g(-1) \\ +g(1)+\cdots+g(N) \end{matrix} + \underbrace{\pi i c_{-1} + \sum_{n=0}^{k} \frac{B_n}{n!} c_{-n}(2\pi i)^n}_{\operatorname{Res}(f,0)}$$

It suffices to show $\lim_{N\to+\infty} \left| \oint_{\gamma_N} f(z)\,dz \right| = 0$:

On γ_N (N large), we have:

$$|f(z)| = \pi |g(z)| \underbrace{|\cot \pi z|}_{\leq C \text{ from HW4 Q2}} \leq \frac{\pi C}{|z|^2} \leq \frac{\pi C}{(N+\frac{1}{2})^2}.$$



$$\therefore \left| \oint_{\gamma_N} f(z)\,dz \right| \leq \frac{\pi C}{(N+\frac{1}{2})^2} \underbrace{4(2N+1)}_{\text{length of } \gamma_N} \longrightarrow 0 \text{ as } N\to+\infty.$$

$$\therefore \lim_{N\to+\infty} \left(\begin{matrix} g(-N)+\cdots+g(-1) \\ +g(1)+\cdots+g(N) \end{matrix} + \underbrace{\pi i c_{-1} + \sum_{n=0}^{k} \frac{B_n}{n!} c_{-n}(2\pi i)^n}_{\operatorname{Res}(f,0)} \right) = 0$$

$$\Rightarrow \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} g(n) = -\pi i c_{-1} - \sum_{n=0}^{k} \frac{B_n}{n!} (2\pi i)^n c_{-n}$$

∎

PROBLEM #3

(a)
$$\frac{c_n}{c_{n-1}} = \frac{\int_0^1 t^n \varphi(t)\,dt}{\int_0^1 t^{n-1}\varphi(t)\,dt} \le \frac{\int_0^1 t^n M\,dt}{\int_0^1 t^{n-1} m\,dt} = \frac{M\cdot\frac{1}{n+1}}{m\cdot\frac{1}{n}} = \frac{M}{m}\cdot\frac{n}{n+1} < \frac{M}{m}.$$

$$\therefore\ c_n < c_{n-1}\frac{M}{m} < c_{n-2}\left(\frac{M}{m}\right)^2 < \cdots < c_1\left(\frac{M}{m}\right)^{n-1}$$

$\forall \varepsilon>0$, we consider the ball $B_{\frac{m}{M}-\varepsilon}(0)$:
(and $\varepsilon<\frac{m}{M}$)

$\forall z\in B_{\frac{m}{M}-\varepsilon}(0)$, we have

$$|c_n z^n| \le c_1\left(\frac{M}{m}\right)^{n-1}\left(\frac{m}{M}-\varepsilon\right)^n$$

$$= \frac{c_1 m}{M}\left(\frac{M}{m}\left(\frac{m}{M}-\varepsilon\right)\right)^n = \frac{c_1 m}{M}\Big(\underbrace{1-\frac{\varepsilon M}{m}}_{<1}\Big)^n$$

$\sum_{n=1}^{\infty}\frac{c_1 m}{M}\left(1-\frac{\varepsilon M}{m}\right)^n$ converges.

By Weierstrass M-test:

$\sum_{n=0}^{\infty}\underbrace{c_n z^n}_{\text{holo. functions}}$ converges uniformly on $B_{\frac{m}{M}-\varepsilon}(0)$ $\forall\varepsilon>0$.

$\Rightarrow \sum_{n=0}^{\infty} c_n z^n$ is holomorphic on $B_{\frac{m}{M}-\varepsilon}(0)$ $\forall \varepsilon>0$.

$\Rightarrow \sum_{n=0}^{\infty} c_n z^n$ is holomorphic on $\bigcup_{\varepsilon>0} B_{\frac{m}{M}-\varepsilon}(0) = B_{\frac{m}{M}}(0)$.

(b)
$\forall t\in[0,1]$, $z\in\Omega$, we note that $1-tz\neq 0$ otherwise we would have $z=\frac{1}{t}\in[1,\infty]$, contradict to $z\in\Omega$.

By extreme value therem,

$\inf\{\underbrace{|1-tz|}_{\text{continuous function}} : (t,z)\in\underbrace{[0,1]\times K}_{\text{compact}}\}$ is achieve at some $(t^*,z^*)\in[0,1]\times K$.

$$\therefore\ \inf_{[0,1]\times K}|1-tz| = |\underbrace{1-t^*z^*}_{\neq 0}| > 0.$$

(c) We need to show:

① $\hat{f}$ is holo. on Ω

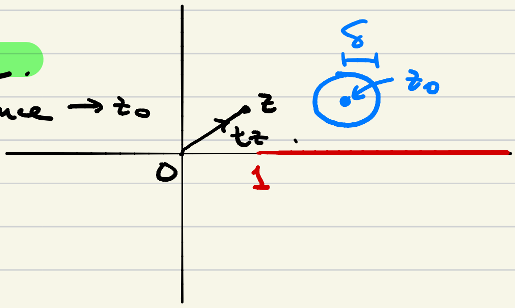
② $\hat{f}(z) = f(z)$ $\quad \forall z \in B_{\frac{m}{M}}(0)$.

To prove ①, we use Morera's Theorem.

First argue $\hat{f}$ is well-defined on Ω: it is because $\forall z \in \Omega$, $1 - tz \neq 0$ $\forall t \in [0,1]$ (otherwise $\underbrace{z = \frac{1}{t}}_{\geq 1}$ $\exists t \in [0,1] \Rightarrow z \notin \Omega$).

Then check that $\hat{f}$ is continuous on Ω.

fix $z_0 \in \Omega$ and let $\{z_j\}$ be a sequence $\to z_0$



Consider $\lim_{j\to\infty} \hat{f}(z_j) = \lim_{j\to\infty} \int_0^1 \frac{\varphi(t)}{1 - tz_j}\, dt$

Let $\delta > 0$ small such that

$$\overline{B_\delta(z_0)} \subset \Omega \quad \left(\text{e.g. take } \delta = \frac{|\mathrm{Im}(z_0)|}{2}\right).$$

then $\varepsilon := \inf_{\underbrace{[0,1]\times\overline{B_\delta(z_0)}}_{\text{compact}}} |1 - tz| \underbrace{> 0}_{\text{by (b)}}$

For large j s.t. $z_j \in B_\delta(z_0)$, we have $\left| \frac{\varphi(t)}{1 - tz_j} \right| \leq \frac{M}{\varepsilon}$

$$\Rightarrow \int_0^1 \left| \frac{\varphi(t)}{1 - tz_j} \right| dt \leq \frac{M}{\varepsilon}$$

LDCT $\Rightarrow$ $\underbrace{\lim_{j\to\infty} \int_0^1 \frac{\varphi(t)}{1 - tz_j}\, dt}_{\hat{f}(z_j)} = \int_0^1 \lim_{j\to\infty} \frac{\varphi(t)}{1 - tz_j}\, dt = \underbrace{\int_0^1 \frac{\varphi(t)}{1 - tz_0}\, dt}_{\hat{f}(z_0)}$.

$\therefore$ $\hat{f}(z_j) \to \hat{f}(z_0)$ $\forall$ sequence $\{z_j\} \to z_0$

$\Rightarrow$ $\hat{f}$ is continuous at z_0.

As z_0 is arbitrary in Ω, $\hat{f}$ is continuous on Ω.

Next show $\oint_T \hat{f}(z)\,dz = 0 \quad \forall$ triangle $T \subset \Omega$:

We argue that $\displaystyle\oint_T \int_0^1 \frac{\varphi(t)}{1-tz}\,dt\,dz = \int_0^1 \overbrace{\oint_T \underbrace{\frac{\varphi(t)}{1-tz}}_{\text{holo. fct. } \forall \text{ fixed } t\in[0,1]} dz}^{0}\,dt = 0$

To justify this, we use Fubini's Theorem.

Note that T is compact,

$\therefore \eta := \inf\{|1-tz| : t\in[0,1],\ z\in T\} > 0$ by (b).

$$\oint_T \int_0^1 \left|\frac{\varphi(t)}{1-tz}\right| dt\,|dz| \le \oint_T \int_0^1 \frac{M}{\underbrace{|1-tz|}_{\ge \eta > 0}}\,dt\,|dz| \le \oint_T\int_0^1 \frac{M}{\eta}\,dt\,|dz|$$

$$= \frac{M}{\eta}\,\text{length}(T) < \infty.$$

It justifies $\oint_T \int_0^1 = \int_0^1 \oint_T$ and hence $\oint_T \hat{f}(z)\,dz = 0 \quad \forall\, T\subset\Omega$ triangle.

Morera's Theorem $\Rightarrow$ $\hat{f}$ is holo. on Ω.

To prove ② $\hat{f}(z) = f(z)$ on $B_{\frac{m}{M}}(0)$:

we consider $\dfrac{1}{1-tz}$, note that $|tz| \le 1\cdot\frac{m}{M} < 1 \quad \forall z\in B_{\frac{m}{M}}(0)$

$$\therefore \frac{1}{1-tz} = \sum_{n=0}^\infty (tz)^n = \sum_{n=0}^\infty t^n z^n.$$

$$\Rightarrow \hat{f}(z) = \int_0^1 \frac{\varphi(t)}{1-tz}\,dt = \int_0^1 \sum_{n=0}^\infty t^n\varphi(t)\,z^n\,dt$$

$$|t^n\varphi(t)z^n| \le \underbrace{M|z|^n}_{\text{indep. of } t} \quad \forall t\in[0,1].$$

$\sum_{n=0}^\infty M\underbrace{|z|}_{|z|<\frac{m}{M}\le 1}{}^n$ converges $\xRightarrow{\text{M-test}}$ $\sum_{n=0}^\infty t^n\varphi(t)z^n$ converges uniformly on $t\in[0,1]$ $\forall$ fixed $z\in B_{\frac{m}{M}}(0)$.

$$\therefore \int_0^1 \sum_{n=0}^\infty t^n\varphi(t)z^n\,dt = \sum_{n=0}^\infty \int_0^1 t^n\varphi(t)z^n\,dt$$

$$= \sum_{n=0}^\infty \underbrace{\left(\int_0^1 t^n\varphi(t)\,dt\right)}_{=c_n} z^n = f(z).$$

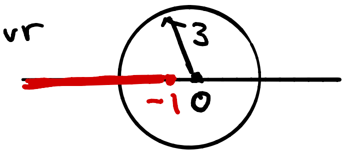
Hence $\hat{f}:\Omega\to\mathbb{C}$ is the analytic continuation of $f: B_{m/M}(0)\to\mathbb{C}$. ☒

{Easy Version}

PROBLEM #1

(a) $\frac{d}{dz}\operatorname{Log}(z+1) = \frac{1}{z+1}$ holds on $\mathbb{C}\setminus(-\infty,-1]$ only.

the contour $|z|=3$



passes through $(-\infty,-1]$

Theorem 3.4 cannot be used.

(b) $\frac{1}{z+1}$ is not complex differentiable at -1 which is enclosed by $|z|=3$. Cauchy-Goursat's Theorem cannot be used.

(c) That Laurent series expansion is for the region $\{|z|<1\}$ only. The contour $|z|=3$ does not lie inside $\{|z|<1\}$.

PROBLEM #2

(a) $$\frac{1}{z^2-4} = \frac{1}{z-2}\cdot\frac{1}{z+2} = \frac{1}{z-2}\cdot\frac{1}{z-2+4} = \frac{1}{z-2}\cdot\frac{1}{4}\cdot\frac{1}{1+\frac{z-2}{4}}$$

Since $\left|\frac{z-2}{4}\right|<1$, we have

$$\frac{1}{1+\frac{z-2}{4}} = 1-\frac{z-2}{4}+\left(\frac{z-2}{4}\right)^2-\left(\frac{z-2}{4}\right)^3+\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^n}{4^n}.$$

$$\Rightarrow \frac{1}{z^2-4} = \frac{1}{4}\cdot\frac{1}{z-2}\sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^n}{4^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^{n-1}}{4^{n+1}}}.$$

$$= \frac{1}{4}\frac{1}{z-2}-\frac{1}{4^2}+\frac{1}{4^3}(z-2)-\cdots \quad \text{on } 0<|z-2|<4.$$

(b) $$\frac{1}{z^2-4} = \frac{1}{z^2}\frac{1}{1-\frac{4}{z^2}} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{4}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{4^n}{z^{2n+2}}.$$

$\uparrow$

$$\left|\frac{4}{z^2}\right| = \frac{4}{|z|^2} < \frac{4}{2^2} = 1$$

From (a), $\operatorname{Res}\left(\frac{1}{z^2-4}, 2\right) = \frac{1}{4}$.

PROBLEM #3

(a) Theorem 3.15 (Liouville's Theorem)

(b) $\forall \alpha \in \mathbb{C}$, $R > |\alpha|$, by 1st-order Cauchy's integral formula:

$$f'(\alpha) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{(z-\alpha)^2} dz$$

$$\Rightarrow \quad |f'(\alpha)| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{|f(z)||dz|}{|z-\alpha|^2}$$

$$\leq \sup_{|z|=R} \left|\frac{f(z)}{z}\right|$$

$$\leq \frac{1}{2\pi} \oint_{|z|=R} \frac{\frac{|f(z)|}{|z|} \underbrace{|z|}_{=R} |dz|}{\left|\underbrace{|z|}_{=R} - |\alpha|\right|^2}$$

$$\leq \frac{1}{\cancel{2\pi}} \sup_{|z|=R} \left|\frac{f(z)}{z}\right| \cdot \frac{R}{(R-|\alpha|)^2} \cdot \underbrace{\cancel{2\pi} R}_{\text{length of } |z|=R.}$$

As $\frac{R^2}{(R-|\alpha|)^2} \to 1$ as $R \to +\infty$,

$\frac{R^2}{(R-|\alpha|^2)}$ is bounded (say by C).

then $|f'(\alpha)| \leq C \sup_{|z|=R} \frac{|f(z)|}{|z|} \to 0$ as $R \to \infty$.

$\therefore f'(\alpha) = 0$

As α is arbitrary, $f' \equiv 0 \Rightarrow f$ is a constant.

PROBLEM #4

(a) For any $r \in (0,1)$, $\forall z \in B_r(0)$, we have

$$|z^n| = |z|^n < \underbrace{r^n}_{M_n}, \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ converges as } r \in (0,1).$$

By Weierstrass M-test, $\sum_{n=0}^{\infty} z^n$ converges uniformly on $B_r(0)$.

(b) $$\left|\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{N} z^n\right| = \left|\sum_{n=N+1}^{\infty} z^n\right| = \left|\frac{z^{N+1}}{1-z}\right|$$

$\sup_{B_1(0)} \left|\frac{z^{N+1}}{1-z}\right| = \infty$ since $\exists$ a sequence $\{1-\frac{1}{n}\}_{n=1}^{\infty} \subset B_1(0)$ s.t. $\lim_{n\to\infty} \frac{(1-\frac{1}{n})^{N+1}}{1-(1-\frac{1}{n})} = \infty$.

$\therefore \sup_{B_1(0)} \left| \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{N} z^n \right| = \infty \not\to 0$ as $N \to +\infty$.

$\therefore \sum_{n=0}^{\infty} z^n$ does not converge uniformly by definition.

PROBLEM #5

$\forall z,\ Re(z) > 0,$

$$\left| \frac{e^{-tz}}{1+t^2} \right| = \frac{e^{-tRe(z)}}{1+t^2} \overset{\substack{t \geq 0 \\ Re(z) > 0}}{\leq} \frac{1}{1+t^2}. \qquad \int_0^{\infty} \frac{1}{1+t^2} dt = [\arctan t]_0^{\infty} = \frac{\pi}{2} < \infty.$$

$\therefore \int_0^{\infty} \frac{e^{-tz}}{1+t^2}$ is well-defined on $\{Re(z) > 0\}$.

To check f is continuous: take a sequence $z_j \to z_0 \in \Omega = \{Re(z) > 0\}$.

We need to justify

$$\lim_{j \to \infty} \int_0^{\infty} \frac{e^{-tz_j}}{1+t^2} dt = \int_0^{\infty} \lim_{j \to \infty} \frac{e^{-tz_j}}{1+t^2} dt.$$

$$\left| \frac{e^{-tz_j}}{1+t^2} \right| \leq \frac{e^{-tRe(z_j)}}{1+t^2} \leq \frac{1}{1+t^2}$$

as $Re(z_j) > 0$, $t \geq 0$.

$$\int_0^{\infty} \frac{1}{1+t^2} dt = \frac{\pi}{2} < \infty.$$

LDCT $\Rightarrow$

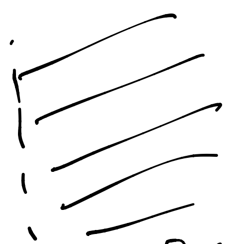
$$\lim_{j \to \infty} f(z_j) = \lim_{j \to \infty} \int_0^{\infty} \frac{e^{-tz_j}}{1+t^2} dt = \int_0^{\infty} \lim_{j \to \infty} \frac{e^{-tz_j}}{1+t^2} dt$$
$$= \int_0^{\infty} \frac{e^{-tz_0}}{1+t^2} dt = f(z_0).$$

$\therefore \lim_{z \to z_0} f(z) = f(z_0) \quad \forall z_0,\ Re(z_0) > 0.$

Next we show $\oint_T f(z)\, dz = 0 \quad \forall$ triangle $T \subset \Omega$.

We claim that

$$\oint_T \int_0^{\infty} \frac{e^{-tz}}{1+t^2} dt\, dz = \int_0^{\infty} \oint_T \frac{e^{-tz}}{1+t^2} dz\, dt \quad \text{---} (*)$$



$Re(z) > 0$.

To justify this, we check that

$$\oint_T \int_0^{\infty} \left| \frac{e^{-tz}}{1+t^2} \right| dt\, |dz| \leq \oint_T \int_0^{\infty} \frac{\overbrace{e^{-tRe(z)}}^{\leq 1 \text{ as } t \geq 0,\ Re(z) > 0}}{1+t^2} dt\, |dz| \leq \oint_T \underbrace{\int_0^{\infty} \frac{1}{1+t^2} dt}_{= \frac{\pi}{2}} |dz| = \frac{\pi}{2} L(T) < \infty.$$

By Fubini's Theorem

$$\oint_T \underbrace{\int_0^\infty \frac{e^{-tz}}{1+t^2}\,dt}_{f(z)}\,dz = \int_0^\infty \overbrace{\oint_T \underbrace{\frac{e^{-tz}}{1+t^2}}_{\text{entire}}\,dz}^{=0 \text{ by Cauchy-Goursat}}\,dt = \int_0^\infty 0\,dt = 0.$$

Moera's Theorem $\Rightarrow$ $f(z)$ is holomorphic on Ω.

PROBLEM #6

$\partial\Omega_R$ encloses only one singularity of $\frac{1}{1+z^2}$: namely i.

$$\therefore \oint_{\partial\Omega_R} \frac{1}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, i\right) = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\lim_{z\to i} (z-i)\cdot\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$$

$\therefore$ i is a simple pole.

$$\underbrace{\oint_{\partial\Omega_R} \frac{1}{1+z^2}\,dz}_{=\pi} = \int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{1+z^2}\,dz + \int_{[-R,R]} \frac{1}{1+x^2}\,dx.$$

$$\left|\int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{1+z^2}\,dz\right| \le \int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{|z|^2-1}\,|dz| = \frac{\pi R}{R^2-1} \to 0 \text{ as } R\to\infty.$$

$$\therefore \quad \pi = \lim_{R\to\infty}\left(\int_{\underset{-R\quad R}{\curvearrowleft}} \frac{1}{1+z^2}\,dz + \int_{-R}^{R} \frac{1}{1+x^2}\,dx\right)$$

$$= 0 + \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \quad \blacksquare$$

PROBLEM #7

(a) $$z^{4023}-1 = (z-1)(z-\omega)\cdots(z-\omega^{4022}) = \prod_{j=0}^{4022}(z-\omega^j)$$

$$\frac{d}{dz}(z^{4023}-1) = \sum_{j=0}^{4022} (z-1)\cdots\underbrace{\frac{d}{dz}(z-\omega^j)}_{=1}\cdots(z-\omega^{4022})$$

$$\frac{\frac{d}{dz}(z^{4023}-1)}{z^{4023}-1} = \sum_{j=0}^{4022} \frac{\cancel{(z-1)}\cdots\cancel{(z-\omega^{j-1})}\cdot 1\cdot\cancel{(z-\omega^{j+1})}\cdots\cancel{(z-\omega^{4022})}}{\cancel{(z-1)}\cdots(z-\omega^j)\cdots\cancel{(z-\omega^{4022})}}$$

only survivor

$$= \sum_{j=0}^{4022}\frac{1}{z-\omega^j}$$

(b) Let $p(z) = a_{4022}z^{4022} + \cdots + a_1 z + a_0 = \sum_{k=0}^{4022} a_k z^k$.

$$\frac{1}{2\pi i}\oint_{|z|=2} p(z)\frac{f'(z)}{f(z)}\,dz \overset{(a)}{=} \frac{1}{2\pi i}\oint_{|z|=2}\sum_{j=0}^{4022}\frac{p(z)}{z-\omega^j}\,dz \overset{\text{Cauchy's integral formula.}}{=} \sum_{j=0}^{4022} p(\omega^j) \quad \text{— }(*)$$

$|z|=2$: enclose all roots of $f(z) := z^{4023}-1$

On the other hand,

$$\frac{1}{2\pi i}\oint_{|z|=2}\frac{p(z)f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{|z|=2}\frac{\sum_{k=0}^{4022} a_k z^k\cdot 4023 z^{4022}}{z^{4023}-1}\,dz$$

$$= \sum_{k=0}^{4022}\frac{1}{2\pi i}\oint_{|z|=2}\frac{4023\, a_k z^{k+4022}}{z^{4023}-1}\,dz$$

By HW2 #7(b)(c), $\oint_{|z|=2}\frac{z^{k+4022}}{z^{4023}-1}\,dz \neq 0$ only when $4023 \mid (k+4022)+1$

For $0\le k\le 4022$, the only survivor is $k=0$.

$$\therefore \frac{1}{2\pi i}\oint_{|z|=2}\frac{p(z)f'(z)}{f(z)} = \frac{1}{2\pi i}\oint_{|z|=2}\frac{a_0\cdot 4023 z^{4022}}{z^{4023}-1}$$

$$= 4023\, a_0\cdot \underset{\uparrow \text{ from } 7(b).}{1} = 4023\, p(0). \quad \text{—}(**)$$

$\therefore$ Combining $(*)$, $(**)$ we get $p(0) = \frac{1}{4023}\sum_{j=0}^{4022} p(\omega^j)$