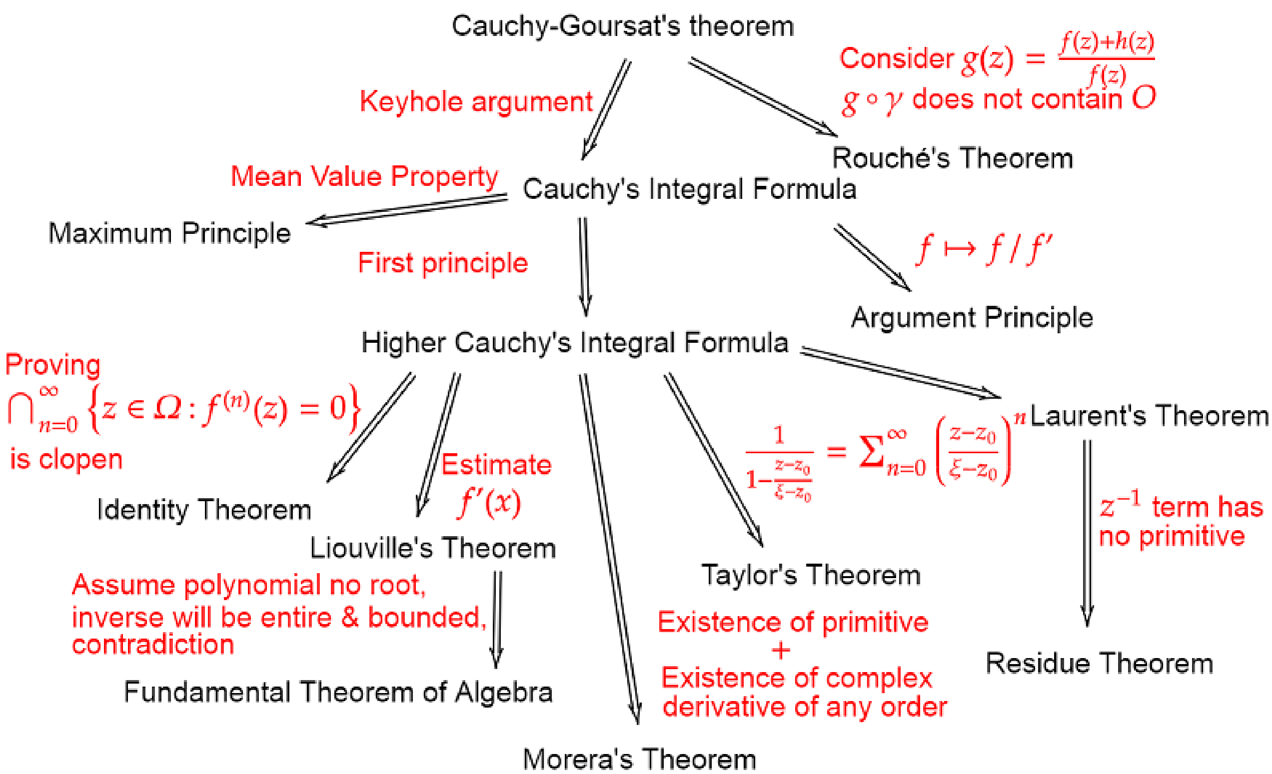
MATH4023
Tutorial 13 - Final Review
May 19, 2020 Notes by Ivan So

1 Review

The reason for why Cauchy-Goursat's theorem is known as the "Fundamental Theorem of Complex Analysis":



2 Problems

1. Using Rouché's theorem, find the number of roots of the equation (counting multiplicities)

$$z^8 - 4z^5 + z^2 - 1 = 0$$

in the disk $|z| < 1$.

Solution: On $|z| = 1$,

$$|z^8 - 4z^5| = |z^3 - 4| \geq 4 - |z^3| = 3, \quad |z^2 - 1| \leq |z^2| + 1 = 2,$$

so $|z^8 - 4z^5| > |z^2 - 1|$ on $|z| = 1$. Therefore the number of roots of $z^8 - 4z^5 + z^2 - 1$ on $|z| < 1$ counting the multiplicity is the same as the number of roots of $z^8 - 4z^5$ on $|z| < 1$. As $z^3 - 4$ has no roots in $|z| < 1$, the number of roots of the concerned equation has to be **5** in the disk $|z| < 1$. ☐

2. By using a branch of $\log(z + i)$ and integrating

$$\frac{\log(z+i)}{z^2+1}dz$$

over the boundary of the upper half disk of radius $R > 0$ centered at 0 as $R \to \infty$, evaluate

$$\int_0^\infty \frac{\log(x^2+1)}{x^2+1}dx$$

and express the answer explicitly as a rational function of π and $\log 2$.

Solution: Select the principal branch to be the positive real axis. Consider the integration over the closed contour of the upper semicircle of radius R, we have:

$$\int_{-R}^{R} \frac{\log(x+i)}{x^2+1}dx + \int_{C_R} \frac{\log(z+i)}{z^2+1}dz = \sum \operatorname{Res}\left(\frac{\log(z+i)}{z^2+1}\right).$$

First consider the residue. The only singularity will be $z = i$. At that point,

$$\operatorname{Res}\left(\frac{\log(z+i)}{z^2+1}, i\right) = 2\pi i \lim_{z\to i}(z-i)\frac{\log(z+i)}{z^2+1} = i\frac{\pi^2}{2} + \pi \log 2.$$

Then over the arc of radius R,

$$\left|\int_{C_R} \frac{\log(z+i)}{z^2+1}dz\right| \leq \pi R \cdot \frac{\log R + \log(1+1/R)}{R^2+1} \xrightarrow{R\to\infty} 0.$$

So from the residue theorem,

$$\begin{aligned}
i\frac{\pi^2}{2} + \pi\log 2 &= \lim_{R\to\infty}\int_{-R}^{R} \frac{\log(x+i)}{x^2+1}dx \\
&= \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}dx \\
&= \int_0^\infty \frac{\log(x+i)+\log(-x+i)}{x^2+1}dx \\
&= \int_0^\infty \frac{\log(x+i)+\log(x-i)+\log(-1)}{x^2+1}dx \\
&= \int_0^\infty \frac{\log(x^2+1)+\log(-1)}{x^2+1}dx \\
&= \int_0^\infty \frac{\log(x^2+1)}{x^2+1}dx + i\pi\int_0^\infty \frac{1}{x^2+1}dx \\
&= \int_0^\infty \frac{\log(x^2+1)}{x^2+1}dx + i\frac{\pi^2}{2}
\end{aligned}$$

$$\Rightarrow \int_0^\infty \frac{\log(x^2+1)}{x^2+1}dx = \pi\log 2.$$

$\square$

3. (Properties of Elliptic Functions) A nonconstant meromorphic function F is said to be **elliptic** if it is doubly periodic (i.e. there exists $\tau \in \mathbb{C}$ such that $F(z+1) = F(z+\tau) = F(z)$). So, the possible values of $F(z)$ is completely determined from the evaluation in the fundamental parallelogram P_0, a compact subset spanned by $1, \tau$. An example of elliptic function is the **Weierstrass $\mathfrak{p}$ function**

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right].$$

We shall prove the special properties for elliptic functions in the following.

(a) An entire doubly periodic function is constant.

(b) The total number of poles of an elliptic function in a fundamental parallelogram is always ≥ 2.

(c) $(\mathfrak{p}')^2 = 4(\mathfrak{p}-\mathfrak{p}(1/2))(\mathfrak{p}-\mathfrak{p}(\tau/2))(\mathfrak{p}-\mathfrak{p}((1+\tau)/2))$.

(d) Every even elliptic function F with period 1 and τ is a rational function of $\mathfrak{p}$.

Solution:

(a) The values of the function is completely determined by the values in the fundamental parallelogram P_0, which is a compact subset of $\mathbb{C}$.
Entire implies continuity over the compact subset P_0, so there exists some z_0 such that $|F(z_0)| = B$ is having the maximum modulus in P_0, further imply that F is bounded by the double periodicity.
So $F(z)$ is a constant (Liouville).

(b) Since linear translation does not change the elliptic properties, we may assume without loss of generality that none of the poles lie on ∂P_0. Consider

$$\begin{aligned}\oint_{\partial P_0} F(z)dz &= \int_0^1 F(z)dz + \int_1^{1+\tau} F(z)dz + \int_{1+\tau}^{\tau} F(z)dz + \int_\tau^0 F(z)dz \\ &= \int_0^1 F(z)dz + \int_0^{\tau} F(z)dz + \int_1^0 F(z)dz + \int_\tau^0 F(z)dz \qquad \text{(double periodicity)}\\ \Rightarrow \sum \mathrm{Res}_{P_0}(F) &= 0 \qquad \text{(residue theorem)}.\end{aligned}$$

It cannot be true that F has no residue in the concerned definition following (a) and definition, so it must be the fact that residues cancel out. It is possible only if there are more than one pole.

(c) The only roots of $F(z) = (\mathfrak{p}(z) - e_1)(\mathfrak{p}(z) - e_2)(\mathfrak{p}(z) - e_3)$ in the fundamental parallelogram are $1/2$, $\tau/2$ and $(1+\tau)/2$. All of them has a multiplicity of 2. Similarly, one can check that $(\mathfrak{p}')^2$ also contains roots of order 2 at the same set of points.
Now consider the poles. $F(z)$ has poles of order 6 at $0, 1, \tau, 1+\tau$. In the fundamental parallelogram. So does $(\mathfrak{p}')^2$. Therefore $(\mathfrak{p}')^2/F$ is an entire and doubly periodic. From (a), it has to be a constant. The constant can be find at places around 0. Around 0,

$$\mathfrak{p}(z) = \frac{1}{z^2} + \text{higher order terms},\ \mathfrak{p}'(z) = \frac{-2}{z^3} + \text{higher order terms},$$

showing the concerned constant to be 4.
Remark: This has a close tie with the studies of *elliptic curves*. One can always show that an elliptic curve over $\mathbb{C}$ in projective space is given by $y^2 = x(x-1)(x-\lambda)$, for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

(d) If F has a zero or pole at 0, it must be even order since F is even. So there exists $m \in \mathbb{Z}$ such that $F\mathfrak{p}^m$ has no zero or pole at the lattice points. We can reduce to the case without zero or poles on the lattice.
Check that if $a_1, \cdots, a_n$ are the zeros of F in the fundamental parallelogram, then $[\mathfrak{p}(z) - \mathfrak{p}(a_1)] \cdots [\mathfrak{p}(z) - \mathfrak{p}(a_n)]$ has exactly the same zeros and also being even. Similar argument applies for poles.
Therefore, F divide by certain rational function generated by Weistrass $\mathfrak{p}$-function will be a doubly periodic and holomorphic function. It has to be a constant by (a). Proving the claim.

$\square$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. Define a complex-valued function F by the following integral over $t \in [0, 1] \subset \mathbb{R}$:

$$F(z) = \int_0^1 t^{z-1} f(t)dt.$$

(a) Show that $F(z)$ is holomorphic on the domain $\Omega := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$.

(b) Suppose further that f is holomorphic on $B_R(0)$ for some $R > 1$. Find the analytic continuation $\hat{F}$ of F on $\mathcal{O} := \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}$.

Solution:

(a) **Step 1: Show that $F(z)$ is well-defined.**

$$|t^{z-1} f(t)| = t^{x-1}|f(t)| \leq Bt^{x-1}dt \qquad (f(t) \text{ is continuous on a compact subset (closed interval)})$$
$$\Rightarrow \left|\int_0^1 t^{z-1} f(t)dt\right| \leq \int_0^1 Bt^{x-1}dt = \frac{B}{x} < \infty$$

So $F(z)$ is well-defined.

Step 2: Define a sequence of function $F_n(z) = \int_{1/n}^1 t^{z-1} f(t)dt$ is continuous with LDCT.
Following the same argument as in Step 1, $F_n(z)$ is well-defined. Define $g_j(t) := t^{z_j-1} f(t)$, where $z_j \to z$ with $\mathrm{Re}\,(z_j) > 0$. We try to prove

$$\lim_{j\to\infty} \int_{1/n}^1 g_j(t)dt = \int_{1/n}^1 \lim_{j\to\infty} g_j(t)dt$$

with LDCT.

(1) $g_j(t) \to t^{z-1} f(t)$ *pointwise.* Follows directly from definition.

(2) $|g_j(t)| \le h(t)$ *for some* $h(t)$ *independent of* j *and integrable over* $[1/n, 1]$.

$$|g_j(t)| = t^{x_j-1} |f(t)| \le \frac{B}{t}, \text{ which is integrable over } [1/n, 1].$$

From this result, we have

$$\lim_{j\to\infty} F_n(z_j) = \lim_{j\to\infty} \int_{1/n}^1 g_j(t)dt = \int_{1/n}^1 \lim_{j\to\infty} g_j(t)dt = F_n(z),$$

this proved continuity.

Step 3: Show that $F_n \rightrightarrows F$ on $\Omega_\epsilon := \{z \in \mathbb{C} : \mathbf{Re}\,(z) > \epsilon > 0\}$, thereby it is continuous over Ω.

$$\begin{aligned} |F_n(z) - F(z)| &= \left| \int_0^{1/n} t^{z-1} f(t)dt \right| \\ &\le \int_0^{1/n} Bt^{x-1} dt \\ &= \frac{B}{xn^x} < \frac{B}{\epsilon n^\epsilon} \end{aligned}$$

upper bound is independent of z, proving uniform convergence. The same argument applies for any $\epsilon > 0$. Since $\Omega = \bigcup_{\epsilon>0} \Omega_\epsilon$, we conclude that $F_n \rightrightarrows F$ over Ω. Since F_n is continuous over Ω, its uniform limit F also has to be continuous over Ω.

Step 4: Show that F_n is holomorphic with Morera's theorem.
Notice that for any triangular contour $T \subset \Omega_\epsilon$, since T is a compact subset, there exists m, M such that $m \le x \le M$. Therefore for $z \in T$

$$\int_{1/n}^1 |t^{z-1} f(t)|dt \le \frac{B}{mn^m} \Rightarrow \oint_T \int_{1/n}^1 |t^{z-1} f(t)|dt\,|dz| \le \frac{B}{mn^m} \cdot \text{length}(T) < \infty$$

Together with the continuity of F_n over Ω, the Fubini-Torelli's theorem concludes

$$\oint_T \int_{1/n}^1 t^{z-1} f(t)dt\, dz = \int_{1/n}^1 \oint_T t^{z-1} f(t)dz\, dt.$$

Notice that $t^{z-1} f(t)$ is holomorphic over Ω, from Cauchy-Goursat's theorem,

$$\oint_T \int_{1/n}^1 t^{z-1} f(t)dt\, dz = 0$$

for any triangle $T \in \Omega$. Proving F_n holomorphic over Ω.

Step 5: Proving F is holomorphic over Ω.
From $F_n \rightrightarrows F$ over Ω,

$$\oint_T F(z)dz = \oint_T \lim_{n\to\infty} F_n(z)dz = \lim_{n\to\infty} \oint_T F_n(z)dz = 0.$$

Proving F holomorphic over Ω.

(b) Since f is holomorphic on $B_R(0)$, from Taylor's theorem, we have

$$\begin{aligned} F(z) &= \int_0^1 t^{z-1} f(t) dt \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^{z+n-1} dt \quad \text{(Taylor)} \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{f^{(n)}(0)}{n!} t^{z+n-1} dt \quad \text{(Integrand is uniformly converging)} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{z+n}. \end{aligned}$$

As $\mathcal{O}$ is a connected domain (any two points can be connected through finitely many line segment), we claim that

$$\widehat{F}(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{z+n}$$

defines an analytic continuation of F over the domain $\mathcal{O} = \mathbb{C} \setminus \{0, -1, -2, \cdots\}$. We will have to check:

(a) $\widehat{F}|_{\Omega} = F$ *on* Ω. Follows from the derivation above.

(b) $\widehat{F}$ *is holomorphic over* $\mathcal{O}$. We first prove that $\hat{F}$ is holomorphic over $\mathcal{O}_\epsilon = \mathbb{C} \setminus \bigcup_{n=0}^{\infty} B_\epsilon(-n)$ for any $\epsilon > 0$. Proposition A.5. has exactly what we need. We will have to check

(i) $\widehat{F}$ *defines a pointwise converging series.* From integral estimation and letting $1 < R' < R$,

$$\left| \frac{f^{(n)}(0)}{n!} \frac{1}{z+n} \right| \leq \frac{\sup_{\partial B_{R'}(0)}}{\epsilon R'^n} =: M_n. \quad \text{(Higher Cauchy's)}$$

Meanwhile, $\sum M_n$ converges. The Weistrass' M-test implies $\widehat{F}$ converge uniformly, so it must also be pointwise converging.

(ii) *Series of derivatives of summand converge uniformly.* Similar argument as in (i), but

$$\left| -\frac{f^{(n)}(0)}{n!} \frac{1}{(z+n)^2} \right| \leq \frac{\sup_{\partial B_{R'}(0)}}{\epsilon^2 R'^n} =: M'_n. \quad \text{(Higher Cauchy's)}$$

Meanwhile, $\sum M'_n$ converges. The Weistrass' M-test implies $\sum_{n=0}^{\infty} \left(\frac{f^{(n)}(0)}{n!} \frac{1}{z+n} \right)'$ converge uniformly on $\mathcal{O}_\epsilon$.

(i) and (ii) together implies $\widehat{F}$ is holomorphic over $\mathcal{O}_\epsilon$ for any $\epsilon > 0$. Therefore $\widehat{F}$ is holomorphic over $\bigcup_{\epsilon > 0} \mathcal{O}_\epsilon = \mathcal{O}$.