Lecture 22

07/05/2020

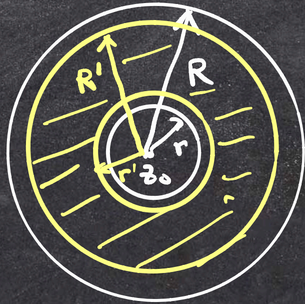
f is holo. on $A_{R,r}(z_0)$ $\quad 0 \le r < R \le +\infty$.

proved $\Rightarrow$ $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ on $A_{R,r}(z_0)$

Claim: $\forall\, r', R'$ s.t.

$0 \le r < r' < R' < R \le +\infty$.

$\sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ converges uniformly on $A_{R',r'}(z_0)$



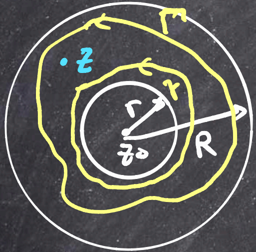
Proofs: ① Weierstrass M-test.

② Estimating the remainders (HW4).

Proof #1:

$$f(\underset{w}{z}) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right) (z - z_0)^n$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\xi) \cdot (\xi - z_0)^{n-1} \, d\xi \right) \frac{1}{(z - z_0)^n}$$



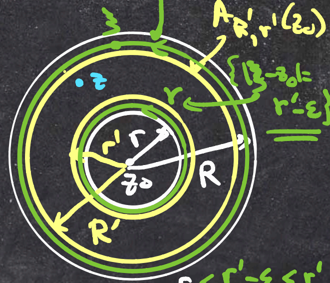
On $z \in A_{R', r'}(z_0)$

$(R' + \varepsilon < R)$

$\Gamma = \{ |\xi - z_0| = R' + \varepsilon \}$

$$\left| \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \cdot (z - z_0)^n \right|$$

$$\le \oint_{\Gamma} \left| \frac{f(\xi)}{\underbrace{(\xi - z_0)^{n+1}}_{|\ \ |^{n} = (R' + \varepsilon)^{n+1}}} \overbrace{(z - z_0)^n}^{|\ \ |^{n} \le (R')^n} \right| |d\xi|$$



$r < r' - \varepsilon < r'$

$$\leq \frac{\max_{\Gamma} |f| \cdot (R')^n}{(R'+\varepsilon)^{n+1}} \cdot 2\pi (R'+\varepsilon)$$

$$= \underbrace{2\pi \max_{\Gamma} |f| \cdot \left(\frac{R'}{R'+\varepsilon}\right)^n}_{M_n} \qquad \forall z \in A_{R',r'}(z_0)$$

$$\sum_{n=1}^{\infty} M_n < \infty \quad \text{since} \quad \underset{\sim}{\frac{R'}{R'+\varepsilon} < 1}.$$

$$\underset{\text{M-test}}{\overset{\Longrightarrow}{\text{Weierstrass}}} \Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right) (z - z_0)^n$$

converges uniformly on $A_{R',r'}(z_0)$.

Similarly
(Exercise)

$$\sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\xi) \cdot (\xi - z_0)^{n-1} \, d\xi \right) \frac{1}{(z - z_0)^n}$$

converges uniformly on $A_{R', r'}(z_0)$.

<u>Cor</u>: $$\oint_{\gamma} \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \, dz = \sum_{n=-\infty}^{\infty} \oint_{\gamma} b_n (z - z_0)^n \, dz.$$

$\forall \gamma \subset A_{R,r}(z_0)$.

Argument Principle.

f holomorphic on $\Omega \setminus S$ ← discrete

open, simply-connected

γ simple closed $\subset \Omega \setminus S$

<u>then</u>:

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz = $$ sum of orders of zeros enclosed by γ

− sum of orders of poles enclosed by γ.

meromorphic on Ω.

$$\lim_{z \to \alpha} (z-\alpha)^m f(z) = L \neq 0 \quad \Rightarrow \quad \mathrm{ord}_f(\alpha) = m.$$

$$\lim_{z \to \beta} \frac{f(z)}{(z-\beta)^m} = M \neq 0 \quad \Rightarrow \quad \mathrm{ord}_f(\beta) = m.$$

e.g. $f(z) = \dfrac{(z-1)^3}{(z+1)^5}$

1 is a zero of f, $\mathrm{ord}_f(1) = 3$

-1 is a pole of f. $(\mathrm{ord}_f(-1) = 5)$

$$\lim_{z \to 1} \frac{f(z)}{(z-1)^3} = \lim_{z \to 1} \frac{1}{(z+1)^5} = \frac{1}{2^5} \neq 0.$$

$$\lim_{z \to -1} (z+1)^5 f(z) = (-2)^3 \neq 0.$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz =$$

0 if 

-5 if 

3 if 

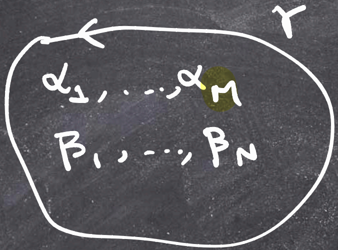
$3 - 5 = -2.$ if 

Sketch of proof of argument principle:

$f(z)$

zeros : $\alpha_1, \dots, \alpha_M$ $\qquad$ $\mathrm{ord}_f(\alpha_i)$

poles : $\beta_1, \dots, \beta_N$ $\qquad$ $\mathrm{ord}_f(\beta_j)$



$$\frac{(z-\beta_1)^{\mathrm{ord}(\beta_1)} \cdots (z-\beta_N)^{\mathrm{ord}(\beta_N)} \quad f(z)}{(z-\alpha_1)^{\mathrm{ord}(\alpha_1)} \cdots (z-\alpha_M)^{\mathrm{ord}(\alpha_M)}} =: F(z).$$

$F(z)$ has no poles and zeros in γ.

$$\Rightarrow f(z) = \frac{\prod_{i=1}^{M} (z-\alpha_i)^{\mathrm{ord}(\alpha_i)}}{\prod_{j=1}^{N} (z-\beta_j)^{\mathrm{ord}(\beta_j)}} F(z)$$

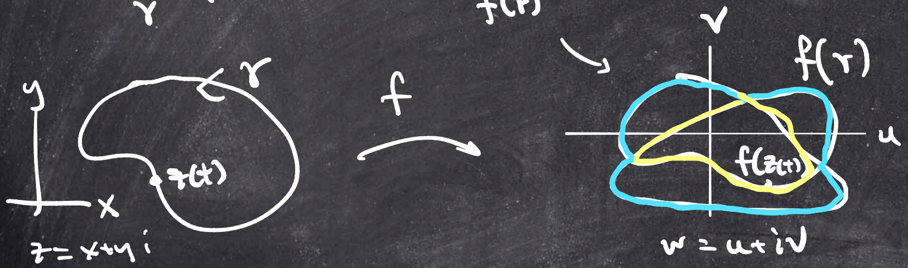
computations

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \sum_{j=1}^{M} \frac{\mathrm{ord}(\alpha_i)}{z-\alpha_i} - \sum_{j=1}^{N} \frac{\mathrm{ord}(\beta_j)}{z-\beta_j} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{holo.}}.$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{M} \mathrm{ord}(\alpha_i) - \sum_{j=1}^{N} \mathrm{ord}(\beta_j) + 0.$$

$\blacksquare$

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i} \oint_{f(\gamma)} \frac{1}{w}\,dw. \quad \text{—— } (*)$$



Proof of (*):

$\gamma: z(t), \quad t \in [a,b].$

$f(\gamma): \quad f(\underbrace{z(t)}_{\gamma})$

$t \in [a,b].$

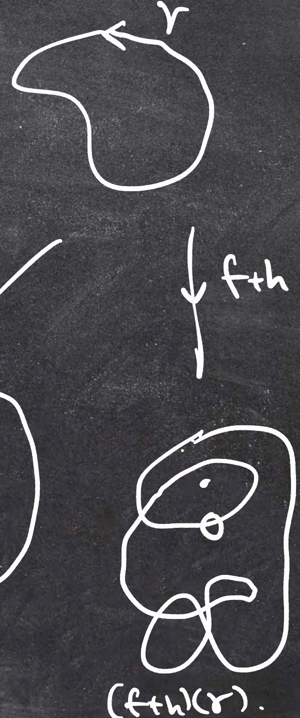
$$\text{LHS} = \oint_\gamma \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(z(t))}{f(z(t))} \cdot z'(t)\,dt.$$

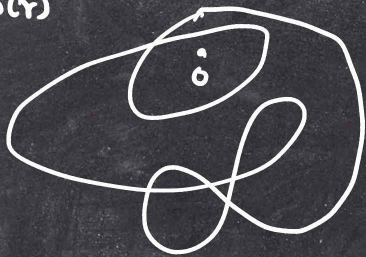
$$\text{RHS} = \oint_{f(\gamma)} \frac{1}{w}\,dw = \int_a^b \frac{1}{f(z(t))} \underbrace{d(f(z(t))}_{= f'(z(t)) \cdot z'(t)\,dt} = \text{LHS}. \qquad \square$$

Rouche's Theorem

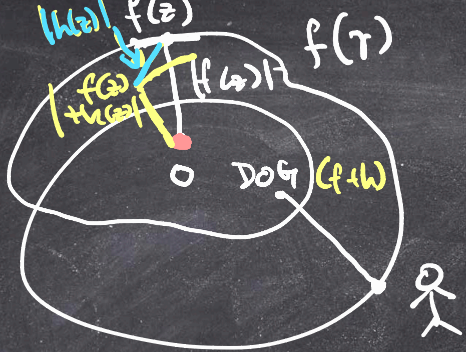
$$\underbrace{|h(z)|}_{\text{small}} < \underbrace{|f(z)|}_{\text{Big}} \quad \text{on } \gamma.$$

$$\Rightarrow \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz}_{\displaystyle \frac{1}{2\pi i}\oint_{f\circ\gamma}\frac{1}{w}\,dw} = \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)+h'(z)}{f(z)+h(z)}\,dz}_{\displaystyle \frac{1}{2\pi i}\oint_{(f+h)(\gamma)}\frac{1}{w}\,dw}$$





$f(\gamma)$



$$|\underbrace{h(z)}| < |f(z)|$$