

Lecture 22

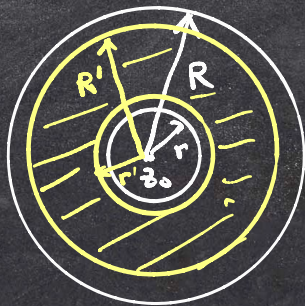
07/05/2020

f is holo. on $A_{R,r}(z_0)$ $0 \leq r < R \leq +\infty$.

proved $\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ on $A_{R,r}(z_0)$

Claim: $\forall r', R'$ s.t.
 $0 \leq r < r' < R' < R \leq +\infty$.

$\sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ converges
uniformly on $A_{R',r'}(z_0)$



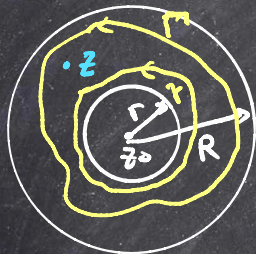
Proofs: ① Weierstrass M-test.

② Estimating the remainders (HW4).

Proof #1:

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right) (z - z_0)^n$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\xi) \cdot (\xi - z_0)^{n-1} d\xi \right) \frac{1}{(z - z_0)^n}$$

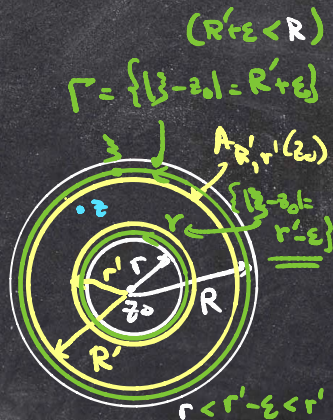


On $z \in A_{R', r'}(z_0)$

$$\left| \oint_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \cdot (z - z_0)^n \right|$$

$$\leq \oint_{\gamma} \left| \frac{f(\xi)}{(\xi - z_0)^{n+1}} \right| |d\xi|$$

$|z - z_0|^n \leq (R')^n$
 $|z - z_0|^{n+1} = (R' + \varepsilon)^{n+1}$



$$\leq \frac{\max_{\Gamma} |f| \cdot (R')^n}{(R' + \varepsilon)^{n+1}} \cdot 2\pi(R' + \varepsilon)$$

$$= \underbrace{2\pi \max_{\Gamma} |f| \cdot \left(\frac{R'}{R' + \varepsilon}\right)^n}_{M_n} \quad \forall z \in A_{R', r'}(z_0)$$

$$\sum_{n=1}^{\infty} M_n < \infty \quad \text{since} \quad \frac{R'}{R' + \varepsilon} < 1.$$

$$\begin{aligned} &\Rightarrow \\ \text{Weierstrass} &\Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right) (z - z_0)^n \\ \text{M-test} & \text{converges uniformly on } A_{R', r'}(z_0). \end{aligned}$$

Similarly
(Exercise)

$$\sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} f(z) \cdot (z - z_0)^{n-1} dz \right) \frac{1}{(z - z_0)^n}$$

converges uniformly on $A_{R', r'}(z_0)$.

Cor: $\oint_{\gamma} \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n dz = \sum_{n=-\infty}^{\infty} \oint_{\gamma} b_n (z - z_0)^n dz.$

$\forall \gamma \subset A_{R, r}(z_0).$

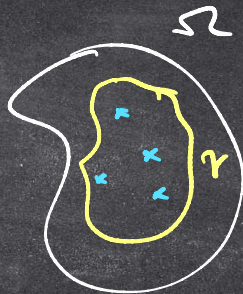


Argument Principle.

f holomorphic on $\Omega \setminus S$ ← discrete
open, simply-connected

γ simple closed

$\Omega \setminus S$



then:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \begin{aligned} &\text{sum of orders} \\ &\text{of zeros enclosed by } \gamma \\ &- \text{sum of orders} \\ &\text{of poles enclosed by } \gamma. \end{aligned}$$

meromorphic.
on Ω .

$$\lim_{z \rightarrow \alpha} (z - \alpha)^m f(z) = L \neq 0 \\ \Rightarrow \text{ord}_f(\alpha) = m.$$

$$\lim_{z \rightarrow \beta} \frac{f(z)}{(z - \beta)^m} = M \neq 0 \\ \Rightarrow \text{ord}_f(\beta) = m.$$

eg. $f(z) = \frac{(z-1)^3}{(z+1)^5}$

1 is a zero of f , $\text{ord}_f(1) = 3$
 -1 is a pole of f .
 ($\text{ord}_f(-1) = 5$)

$$\lim_{z \rightarrow 1} \frac{f(z)}{(z-1)^3} = \lim_{z \rightarrow 1} \frac{1}{(z+1)^5} = \frac{1}{2^5} \neq 0.$$

$$\lim_{z \rightarrow -1} (z+1)^5 f(z) = (-2)^3 \neq 0.$$

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \begin{cases} 0 & \text{if } \begin{matrix} -i & i \end{matrix} \\ -5 & \text{if } \begin{matrix} -i & i \end{matrix} \\ 3 & \text{if } \begin{matrix} -i & i \end{matrix} \\ 3-5 = -2 & \text{if } \begin{matrix} -i & i \end{matrix} \end{cases}$$

(Note: The cases above correspond to different contours γ in the complex plane, indicated by yellow circles and arrows in the original image. The first case is for a contour enclosing neither pole. The second, third, and fourth cases are for contours enclosing one or both poles, with the residue being the sum of the residues of the enclosed poles.)

Sketch of proof of argument principle:

$f(z)$

zeros :

$\alpha_1, \dots, \alpha_m$

$\text{ord}_f(\alpha_i)$

poles :

$\beta_1, \dots, \beta_N$

$\text{ord}_f(\beta_j)$

$$\frac{(z-\beta_1)^{\text{ord}(\beta_1)} \cdots (z-\beta_N)^{\text{ord}(\beta_N)} f(z)}{(z-\alpha_1)^{\text{ord}(\alpha_1)} \cdots (z-\alpha_m)^{\text{ord}(\alpha_m)}} =: F(z).$$

$$\Rightarrow f(z) = \frac{\prod_{i=1}^m (z-\alpha_i)^{\text{ord}(\alpha_i)}}{\prod_{j=1}^N (z-\beta_j)^{\text{ord}(\beta_j)}} F(z)$$

has no poles
and zeros
in γ .

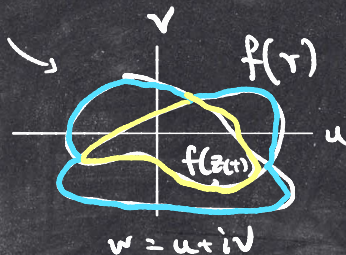
computations

$$\Rightarrow \frac{f'(z)}{f(z)} = \sum_{i=1}^n \frac{\text{ord}(\alpha_i)}{z - \alpha_i} - \sum_{j=1}^N \frac{\text{ord}(\beta_j)}{z - \beta_j} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{holo.}}$$

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^n \text{ord}(\alpha_i) - \sum_{j=1}^N \text{ord}(\beta_j) + 0.$$

□

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{f(\gamma)} \frac{1}{w} dw. \quad (*)$$



Proof of (4):

$$\gamma: z(t), \quad t \in [a, b].$$

$$f(\gamma): \underbrace{f(z(t))}_{\gamma} \\ t \in [a, b].$$

$$\text{LHS} = \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(z(t))}{f(z(t))} \cdot z'(t) dt.$$

$$\begin{aligned} \text{RHS} = \oint_{f(\gamma)} \frac{1}{w} dw &= \int_a^b \frac{1}{f(z(t))} \frac{d(f(z(t)))}{dt} dt = \text{LHS}. \\ &= f'(z(t)) \cdot z'(t) dt \end{aligned}$$

□

Rouche's Theorem

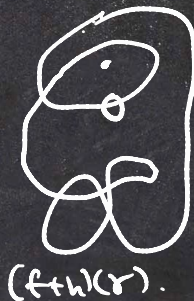
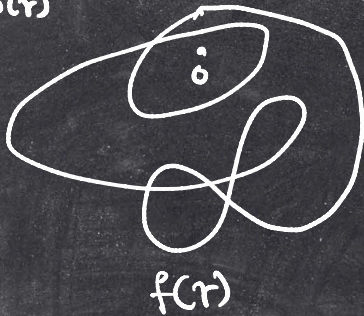
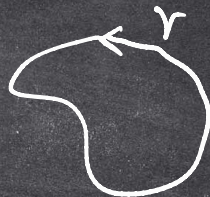
$$|\underbrace{h(z)}_{\text{small}}| < |\underbrace{f(z)}_{\text{Big}}| \quad \text{on } \gamma.$$

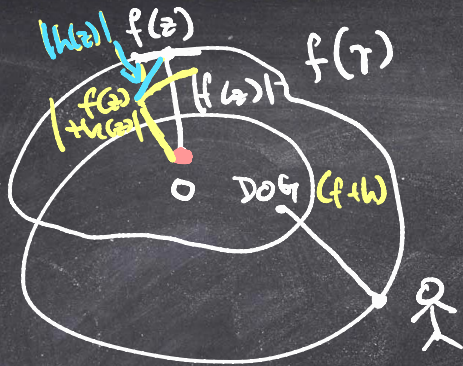
$$\Rightarrow \underbrace{\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz}_{\text{"}} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z) + h'(z)}{f(z) + h(z)} dz$$

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w} dw$$

for

$$\frac{1}{2\pi i} \oint_{(f+h)(\gamma)} \frac{1}{w} dw$$





$$\underline{|h(z)|} < |f(z)|$$