MATH4023
Tutorial 11 - Residue Calculus
May 5/7, 2020 Notes by Ivan So

1 Review

Definition 1.1. (singularity) A point z_0 for a function f is said to be a (an):

- **isolated singularity** if there exists ϵ such that f is holomorphic on $A_{\epsilon,0}(z_0)$;
- **removable singularity** if f can be redefined such that f is holomorphic over Ω;
- **essential singularity** if the c_{-n} is non-zero for infinitely many $n \in \mathbb{N}$.

Definition 1.2. (zero and pole) $z_0 \in \mathbb{C}$ is a **zero** of a *holomorphic function* f if $f(z_0) = 0$. $z_\infty \in \mathbb{C}$ is **pole** of a *holomorphic function* f is a point such that $1/f(z_0) = 0$ and $1/f$ is *holomorphic* in a neighborhood of z_∞.

Proposition 1.3. If f has a pole of order n at z_0, then there exists some *holomorphic function* $G(z)$ in a neighborhood of z_∞ such that

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z).$$

Definition 1.4. (residue) Given a pole z_∞ of a function f, the **residue** is defined to be a_{-1}, the coefficient of $(z-z_0)^{-1}$.

Definition 1.5. (order of pole) Suppose z_0 is a pole of f. Then the **order** of z_0 as a pole is the largest nonnegative integer such that c_{-k} of the Laurent series based at z_0 is nonzero.

Proposition 1.6. Suppose f has a *pole* at z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z).$$

Theorem 1.7. (residue theorem) Suppose f is a *holomorphic* in a open subset $\Omega \subset \mathbb{C}$ containing a circle C and Int$C$, except for a pole z_0 in Int$C$. Then

$$\int_C f(z)dz = 2\pi i \cdot \mathrm{Res}(f, z_0).$$

Proof sketch:

Corollary 1.8. Suppose f is a *holomorphic* in a open subset $\Omega \subset \mathbb{C}$ containing a circle C and Int$C$, except for a poles $z_1, \cdots, z_m$ in Int$C$. Then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f, z_j).$$

Application: Evaluation of real integral.

Definition 1.9. (meromorphic function) Let $\Omega \subset \mathbb{C}$ be an open subset. f is said to be **meromorphic** if f is holomorphic on $\Omega \setminus S$, where S is a discrete set of isolated singularities in Ω which are all poles of f.

Theorem 1.10. (argument principle) Suppose f is *meromorphic* in an open set Ω containing a circle C and its interior. If f has no zero and no pole on C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \sum_{j=1}^{M} \mathrm{ord}(\alpha_j) - \sum_{k=1}^{N} \mathrm{ord}(\beta_k),$$

where $\{\alpha_1, \cdots, \alpha_M\}$ and $\{\beta_1, \cdots, \beta_N\}$ are respectively zeros and poles of f and ord means the order of zeros and poles.

Proof sketch:

- A meromorphic function can be written as

$$f(z) = \frac{\prod_{j=1}^{M}(z-\alpha_j)^{\mathrm{ord}(\alpha_j)}}{\prod_{j=1}^{N}(z-\beta)^{\mathrm{ord}(\beta_j)}}F(z), \text{ where } F(z) \text{ is nonvanishing and holomorphic over } \Omega.$$

- Recall that $f \mapsto f/f'$ sends product of functions to sum of function.
- Apply Cauchy's integral formula to get the concerned evaluation.

Theorem 1.11. (Rouché's theorem) Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |h(z)|$ for all $z \in C$, then

$$\oint_C \frac{f'(z)+h'(z)}{f(z)+h(z)}dz = \oint_C \frac{f'(z)}{f(z)}dz.$$

Proof sketch:

- Consider the function $g(z) = \frac{f(z)+h(z)}{f(z)}$, then the function maps γ into a curve in $B_1(1)$.
- $g \circ \gamma$ does not enclose zero, implying

$$\oint_{g\circ\gamma} \frac{1}{w}dw = 0.$$

  The resulting statement just follow from recalling the definition of g.

2 Problems

1. **True or False**

   (a) The function $f(z) = z/z$ has an isolated singularity at $z = 0$.

   (b) The function $f(z) = \log(1 + z) - \log z$ has a essential singularity at $z = 0$.

2. (a) Let $z_1, \cdots, z_n$ be distinct complex numbers. Determine the explicit partial fraction decomposition of $\frac{1}{(z-z_1)\cdots(z-z_n)}$.

   (b) Let $P(z)$ be a polynomial of degree $\leq n - 1$ and $a_1, \cdots, a_n$ be distinct complex numbers. Assume there is a partial fraction decomposition of the form

$$\frac{P(z)}{(z - a_1)\cdots(z - a_n)} = \frac{c_1}{z - a_1} + \cdots + \frac{c_n}{z - a_n}.$$

   Prove that

$$c_1 = \frac{P(a_1)}{(a_1 - a_2)\cdots(a_1 - a_n)},$$

   and find similarly other coefficients c_j.

3. (Open Mapping Theorem) Prove that if f is holomorphic and non-constant in a region Ω, then f is open (meaning the image of f is open).

4. Let f be meromorphic on $\mathbb{C}$ but not entire. Let $g(z) = e^{f(z)}$. Show that g is not meromorphic on $\mathbb{C}$.

5. Let $\{z_n\}$ be a sequence of *distinct* complex numbers such that $\sum \frac{1}{|z_n|^3}$ converges. (a) Prove that the series

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right)$$

defines a *meromorphic* function on $\mathbb{C}$ on $B_R(0)$. (b) Where are the poles of this function?