MATH4023
Tutorial 11 - Residue Calculus
May 5/7, 2020 Notes by Ivan So

1 Review

Definition 1.1. (singularity) A point z_0 for a function f is said to be a (an):

- **isolated singularity** if there exists ϵ such that f is holomorphic on $A_{\epsilon,0}(z_0)$;
- **removable singularity** if f can be redefined such that f is holomorphic over Ω;
- **essential singularity** if the c_{-n} is non-zero for infinitely many $n \in \mathbb{N}$.

Definition 1.2. (zero and pole) $z_0 \in \mathbb{C}$ is a **zero** of a *holomorphic function* f if $f(z_0) = 0$. $z_\infty \in \mathbb{C}$ is **pole** of a *holomorphic function* f is a point such that $1/f(z_0) = 0$ and $1/f$ is *holomorphic* in a neighborhood of z_∞.

Proposition 1.3. If f has a pole of order n at z_0, then there exists some *holomorphic function* $G(z)$ in a neighborhood of z_∞ such that

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z).$$

Definition 1.4. (residue) Given a pole z_∞ of a function f, the **residue** is defined to be a_{-1}, the coefficient of $(z-z_0)^{-1}$.

Definition 1.5. (order of pole) Suppose z_0 is a pole of f. Then the **order** of z_0 as a pole is the largest nonnegative integer such that c_{-k} of the Laurent series based at z_0 is nonzero.

Proposition 1.6. Suppose f has a *pole* at z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

Theorem 1.7. (residue theorem) Suppose f is a *holomorphic* in a open subset $\Omega \subset \mathbb{C}$ containing a circle C and Int$C$, except for a pole z_0 in Int$C$. Then

$$\int_C f(z)dz = 2\pi i \cdot \mathrm{Res}(f, z_0).$$

Proof sketch:

Corollary 1.8. Suppose f is a *holomorphic* in a open subset $\Omega \subset \mathbb{C}$ containing a circle C and Int$C$, except for a poles $z_1, \cdots, z_m$ in Int$C$. Then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f, z_j).$$

Application: Evaluation of real integral.

Definition 1.9. (meromorphic function) Let $\Omega \subset \mathbb{C}$ be an open subset. f is said to be **meromorphic** if f is holomorphic on $\Omega \setminus S$, where S is a discrete set of isolated singularities in Ω which are all poles of f.

Theorem 1.10. (argument principle) Suppose f is *meromorphic* in an open set Ω containing a circle C and its interior. If f has no zero and no pole on C, then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{M} \mathrm{ord}(\alpha_j) - \sum_{k=1}^{N} \mathrm{ord}(\beta_k),$$

where $\{\alpha_1, \cdots, \alpha_M\}$ and $\{\beta_1, \cdots, \beta_N\}$ are respectively zeros and poles of f and ord means the order of zeros and poles.

Proof sketch:

- A meromorphic function can be written as

$$f(z) = \frac{\prod_{j=1}^{M}(z-\alpha_j)^{\mathrm{ord}(\alpha_j)}}{\prod_{j=1}^{N}(z-\beta)^{\mathrm{ord}(\beta_j)}}F(z), \text{ where } F(z) \text{ is nonvanishing and holomorphic over } \Omega.$$

- Recall that $f \mapsto f/f'$ sends product of functions to sum of function.
- Apply Cauchy's integral formula to get the concerned evaluation.

Theorem 1.11. (Rouché's theorem) Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |h(z)|$ for all $z \in C$, then

$$\oint_C \frac{f'(z)+h'(z)}{f(z)+h(z)}dz = \oint_C \frac{f'(z)}{f(z)}dz.$$

Proof sketch:

- Consider the function $g(z) = \frac{f(z)+h(z)}{f(z)}$, then the function maps γ into a curve in $B_1(1)$.
- $g \circ \gamma$ does not enclose zero, implying

$$\oint_{g\circ\gamma} \frac{1}{w}dw = 0.$$

The resulting statement just follow from recalling the definition of g.

2 Problems

1. **True or False**

(a) The function $f(z) = z/z$ has an isolated singularity at $z = 0$.
False. Redefinition of $f(z) = z/z = 1$, which has no singularity.

(b) The function $f(z) = \log(1+z) - \log z$ has a essential singularity at $z = 0$.
True. $f(z) = \log(1+z) - \log z = \log\left(1+\frac{1}{z}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{z}\right)^n$.

2. (a) Let $z_1, \cdots, z_n$ be distinct complex numbers. Determine the explicit partial fraction decomposition of $\frac{1}{(z-z_1)\cdots(z-z_n)}$.

(b) Let $P(z)$ be a polynomial of degree $\leq n-1$ and $a_1, \cdots, a_n$ be distinct complex numbers. Assume there is a partial fraction decomposition of the form

$$\frac{P(z)}{(z-a_1)\cdots(z-a_n)} = \frac{c_1}{z-a_1} + \cdots + \frac{c_n}{z-a_n}.$$

Prove that

$$c_1 = \frac{P(a_1)}{(a_1-a_2)\cdots(a_1-a_n)},$$

and find similarly other coefficients c_j.

Solution:

(a) In partial fraction expression,

$$f(z) := \frac{1}{(z-z_1)\cdots(z-z_n)} = \frac{a_1}{z-z_1} + \cdots + \frac{a_n}{z-z_n}.$$

Pick a sufficiently small ϵ such that $B_\epsilon(z_i)$ contains no z_j except for $i = j$, then over $B_\epsilon(z_i)$, $\frac{a_j}{z-z_j}$ is holomorphic for $i \neq j$. Therefore,

$$\int_{\overline{B}_\epsilon(z_i)} f(z)dz = \int_{\overline{B}_\epsilon(z_i)} \left(\frac{a_1}{z-z_1} + \cdots + \frac{a_n}{z-z_n}\right)dz$$

$$\mathrm{Res}(f, z_i) = a_i \qquad \text{(Cauchy's integral formula and Residue theorem)}$$

From the residue formula, one can determine easily that

$$\mathrm{Res}(f, z_i) = \prod_{j=1, j\neq i}^{n} \frac{1}{z_i - z_j} = a_i.$$

(b) Note that $P(z)$ is entire. The principle behind is the same as (a).

$\square$

3. (Open Mapping Theorem) Prove that if f is holomorphic and non-constant in a region Ω, then f is open (meaning the image of f is open).

Solution: Let $w_0 \in f(\Omega)$, say $w_0 = f(z_0)$. We want to prove that all points near w_0 must also belongs to the image of f. Define $g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) =: F(z) + G(z)$.
Pick $\delta > 0$ such that $\overline{B}_\delta(z_0) \subset \Omega$ and $f(z) \neq w_0$ on $\partial\overline{B}_\delta(z_0)$. Then, select $\epsilon > 0$ such that $|f(z) - w_0| \geq 0$ on $\partial\overline{B}_\delta(z_0)$. If $|w - w_0| < \epsilon$, then $|F(z)| > |G(z)|$ on the circle $\partial\overline{B}_\delta(z_0)$.
By Rouché's theorem, $g = F + G$ has zero inside $\partial\overline{B}_\delta(z_0)$ since F has a zero inside $\partial\overline{B}_\delta(z_0)$. $\square$

4. Let f be meromorphic on $\mathbb{C}$ but not entire. Let $g(z) = e^{f(z)}$. Show that g is not meromorphic on $\mathbb{C}$.

Solution: f is meromorphic but not entire implies there exists at least a pole at certain $z_0 \in \mathbb{C}$. In some neighborhood of z_0, we can write

$$(z - z_0)^m f(z) = p(z) + (z - z_0)^m h(z),$$

where h is holomorphic and p is a polynomial of degree $< m$. Then,

$$f(z) = \frac{p(z)}{(z - z_0)^m} + h(z),$$

implying

$$e^{f(z)} = e^{\frac{p(z)}{(z-z_0)^m}} e^{h(z)}.$$

One can easily check that $e^{h(z)}$ is holomorphic. However, $e^{\frac{p(z)}{(z-z_0)^m}}$ is having an essential singularity at z_0. Therefore $e^{f(z)}$ cannnot be meromorphic (notice that essential singularity is not a type of pole). $\square$

5. Let $\{z_n\}$ be a sequence of *distinct* complex numbers such that $\sum \frac{1}{|z_n|^3}$ converges. (a) Prove that the series

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right)$$

defines a *meromorphic* function on $\mathbb{C}$ on $B_R(0)$. (b) Where are the poles of this function?

Solution: Fix an $R > 0$, then the hypothesis $\sum \frac{1}{|z_n|^3}$ converges implies $|z_n| \to \infty$ as $n \to \infty$. Therefore, there exists only finitely many n such that $z_n \in B_R(0)$.
The above statement implies there exists N such that $n > N$, $|z_n| > 2R$. Rewrite

$$f(z) = \sum_{n=1}^{N} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right) + \sum_{n=N+1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right),$$

then $z_n \in B_R(0)$ are the poles of order 2. What remains is to show that $f(z)$ is meromorphic. So we will have to show that the second term in $f(z)$ is holomorphic. We have the estimation

$$\begin{aligned}
\left| \frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right| = \left| \frac{-z^2 + 2zz_n}{z_n^2 (z - z_n)^2} \right| &= \frac{1}{|z_n|^3} \frac{\left| -\frac{z^2}{z_n} + 2z \right|}{\left| \frac{z}{z_n} - 1 \right|^2} \\
&\leq \frac{1}{|z_n|^3} \frac{\frac{R^2}{2R} + 2R}{\left(1 - \frac{R}{2R}\right)} \\
&\leq \frac{B}{|z_n|^3}, \quad \text{(for some constant } B\text{)}.
\end{aligned}$$

While the second sum do not contains poles, a quotient of polynomials define a holomorphic function, proving $f(z)$ as a whole is meromorphic. $\square$