MATH4023
Tutorial 10 - Taylor Series and Laurent Series
April 28/30, 2020 Notes by Ivan So

1 Review

Theorem 1.1. (Taylor's theorem for holomorphic function) Suppose f is *holomorphic* in an open disc $B_R(z_0)$, then for any $z \in B_R(z_0)$, the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges *pointwise* to $f(z)$. This is known as the **Taylor's series** of the holomorphic function f at z_0.

Proof sketch: Let $R' = R - \epsilon$ for some $\epsilon > 0$ such that $z \in B_{R'}(z_0)$. Consider $f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R'} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R'} \frac{f(\zeta)}{\zeta - z_0}\frac{1}{1-\frac{z-z_0}{\zeta - z_0}} d\zeta$.

Corollary 1.2. (Taylor remainder for holomorphic function) Let f satisfying the conditions as in the Taylor's theorem, then for any closed curve $\gamma \subset B_R(z_0)$ enclosing z and z_0

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \underbrace{\frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z}\left(\frac{z-z_0}{\zeta - z_0}\right)^N d\zeta}_{=:R_N(z),\text{ the remainder term}}$$

Proof sketch: Similar to the proof of Taylor's theorem, but notice that $\frac{1-\left(\frac{z-z_0}{\zeta-z_0}\right)^N}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)} = \sum_{n=1}^{N-1}\frac{1}{(z-z_0)^n}$.

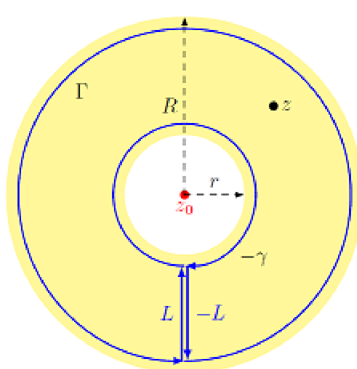
Theorem 1.3. (Laurent) Suppose f is a *holomorphic* function defined on the *annulus* $A_{r,R}(z_0)$ for $r, R \in [0,\infty]$ and $r < R$, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

for some complex numbers c_n. The series is known as the **Laurent series** of the function f at z_0.

Proof sketch:

- Consider the keyhole domain



- In the defined domain we have

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\oint_{\gamma}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\oint_{\Gamma}\frac{1}{\zeta - z_0}\frac{f(\zeta)}{1-\frac{z-z_0}{\zeta - z_0}}d\zeta + \frac{1}{2\pi i}\oint_{\gamma}\frac{1}{z-z_0}\frac{f(\zeta)}{1-\frac{\zeta-z_0}{z-z_0}}d\zeta.$$

  Expand the series.

Corollary 1.4. (Laurent remainder) Let f be a function meeting the requirement of Laurent's theorem, then for each $N \in \mathbb{Z}^+$ and $z \in A_{R,r}(z_0)$,

$$f(z) = \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\zeta)(\zeta - z_0)^{n-1} d\zeta \right) \frac{1}{(z - z_0)^n} + \underbrace{\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{z - \zeta} \left(\frac{\zeta - z_0}{z - z_0} \right)^N d\zeta}_{=: r_N(z)}$$

$$+ \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \underbrace{\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{z - \zeta} \left(\frac{z - z_0}{\zeta - z_0} \right)^N d\zeta}_{=: R_N(z)}$$

where Γ and γ are any pair of circles in $A_{R,r}(z_0)$ centered at z_0 such that z is in between Γ and γ.

Proof sketch: Similar to the proof of Taylor's theorem through considering a geometric sum.

2 Problems

1. **True or False**

   (a) Laurent's theorem is a consequence of Cauchy-Goursat's theorem.
   True. Notice the use of Cauchy integral formula. □

   (b) Suppose $f(z)$ is defined on $B_r(z_0)$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then $f(z)$ is holomorphic.
   False. The converse of Taylor's theorem is not true in general. We can reduce our consideration to a real valued function. Consider the function

$$f(x) = \begin{cases} \sin(e^{1/x^4})e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   Since $\lim_{x\to 0} f(x) = 0$, $f(x) = o(x)$ for a sum of terms of power > 1. But check that the derivative is not continuous at $x = 0$. □

2. Suppose $|f(z)| \leq A + B|z|^k$ and f is entire. Show that all the coefficients c_j, $j > k$ in it's power series expansion are 0.

   Solution: If f is entire, consideration of the series expansion at the origin will be suffice. Under such circumstance.

$$\begin{aligned} c_j &= \frac{1}{j!}f^{(j)}(0) = \frac{1}{2\pi i}\oint_{C_R}\frac{f(\xi)}{\xi^{j+1}}d\xi \quad \text{(Higher Order Cauchy Integral Formula)} \\ \Rightarrow |c_j| &\leq \frac{1}{2\pi}\cdot 2\pi R \sup_{z\in C_R}\left|\frac{f(z)}{R^{j+1}}\right| \quad \text{(Integral Approximation)} \\ &\leq \frac{A}{R^n} + \frac{A}{R^{j-k}} \end{aligned}$$

   Since we can take R to be arbitrarily large, from the last inequality above, this implies $c_j = 0$ for $j > k$. □

3. (Weierstrass theorem) Prove that if f is holomorphic on $\mathbb{C}\setminus\{z_1, \cdots, z_N\}$, then there are $N+1$ entire functions $f_0, \cdots, f_N$ such that

$$f(z) = f_0(z) + f_1(1/(z-z_1)) + \cdots + f_N(1/(z-z_N)).$$

   Solution: Consider the Laurent expansion at one of the singular point say z_1, then there exists $R_1 > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_1)^n + \sum_{n=1}^{\infty} b_n(z-z_1)^{-n}.$$

   The series $\sum_{n=1}^{\infty} b_n(z-z_1)^{-n}$ converges for $z \neq z_1$. This fact implies,

$$f_1(z) = \sum_{n=1}^{\infty} b_n z^n$$

   is entire. The function

$$F_1(z) = f(z) - f_1(1/(z-z_1))$$

   has a removable singularity at z_1. If $N = 1$, the function is entire and the result is proved.
   Assume $N > 1$, reiterate the argument, and we will obtain entire function f_2 such that z_2 is a removable singularity of the function

$$F_2(z) = F_1(z) - f_2(1/1(z-z_2)) = f(z) - [f_1(1/(z-z_1)) + f_2(1/(z-z_2))].$$

   Reiterating this argument a finite number of times, we obtain the results. □

4. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an *entire* function? Justify your answer.

Solution: Let $P(z) = a_d z^d + \cdots + a_0$, then if we write Laurent expansion of $P(z)e^{1/z}$ as $\sum_{n=-\infty}^{\infty} c_n z^n$, we have

$$c_{-N} = \frac{a_0}{N!} + \frac{a_1}{(N+1)!} + \cdots + \frac{a_d}{(N+d)!}.$$

Let r be the smallest nonnegative integer such that $a_r \neq 0$, then

$$c_{-N} = \frac{1}{(N+r)!}\left(a_r + \frac{a_{r+1}}{N+r+1} + \cdots + \frac{a_d}{(N+r+1)\cdots(N+d)}\right),$$

which is nonzero for any large N. Therefore no polynomial such that $P(z)e^{1/z}$ is entire. □

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, whose Taylor's series about 0 is given by:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

(a) Show that for any integer $n \geq 1$ and $r \in \mathbb{R}^{>0}$, we have:

$$\int_0^{2\pi} f(re^{i\theta}) \sin n\theta d\theta = i\pi a_n r^n.$$

(b) Two real numbers α and β are said to have the same sign if they are both positive or both negative or both zero. Suppose further that for each $z \in \mathbb{C}$, $\mathrm{Im}(f(z))$ and $\mathrm{Im}(z)$ always have the same sign.

(i) Show that $a_n \in \mathbb{R}$ for any $n \geq 0$.

(ii) Using the result of (i), show that

$$f(z) = a_0 + a_1 z$$

for any $z \in \mathbb{C}$. [Hint: You can use without proofs that $n\sin\theta + \sin n\theta$, $n\sin\theta - \sin n\theta$ and $\sin\theta$ all must have the same sign for any integer $n \geq 2$ and any $\theta \in [0, 2\pi]$.]

Solution:

(a) Define

$$I := \int_0^{2\pi} f(re^{i\theta}) \sin n\theta d\theta = \int_0^{2\pi} f(re^{i\theta}) \left(\frac{e^{in\theta} - e^{-in\theta}}{2i}\right) d\theta.$$

Let $z = re^{i\theta}$, $\theta \in [0, 2\pi]$, then $dz = ire^{i\theta}d\theta = izd\theta$. This implies:

$$\begin{aligned}
I &= \oint_{|z|=r} f(z) \cdot \frac{\left(\frac{z}{r}\right)^n - \left(\frac{r}{z}\right)^n}{2i} \cdot \frac{1}{iz} dz \\
&= -\oint_{|z|=r} \frac{f(z)}{2z}\left(\frac{z^n}{r^n} - \frac{r^n}{z^n}\right) dz \\
&= \oint_{|z|=r} \left(\frac{f(z)r^n}{2z^{n+1}} - \underbrace{\frac{f(z)z^{n-1}}{2r^n}}_{\text{entire for } n\geq 1}\right) dz \\
&= \frac{r^n}{2} \cdot \frac{2\pi i}{n!} f^{(n)}(0) \qquad \text{(Higher Cauchy's Integral Formula)} \\
&= i\pi r^n a_n, \qquad \text{for } n \geq 1
\end{aligned}$$

(b) (i) $\mathrm{Im}(f(z))$ and $\mathrm{Im}(z)$ have the same sign implies $\mathrm{Im}(f(x+0i))$ and $\mathrm{Im}(x+0i) = 0$ have the same sign. Therefore $\mathrm{Im}(f) = 0$ for any $x \in \mathbb{R}$. Then

$$f'(x+0i) = \left.\frac{\partial f}{\partial x}\right|_{(x,0)} = \left.\frac{\partial u}{\partial x}\right|_{(x,0)} \qquad (\text{since } v(x,0) \equiv 0)$$

$$\Rightarrow \mathrm{Im}(f'(x+0i)) = 0$$

Inductively, one can show $\mathrm{Im}(f^{(n)}(x+0i)) = 0$. This concluded the claim.

(ii) Using the hint, $n\sin\theta \pm \sin n\theta$ and $\sin\theta$ have the same sign (we denote by $\sim$ for the notion of same sign in the following), therefore

$$\begin{aligned}
&\int_0^{2\pi} \mathrm{Im}(f(re^{i\theta}))(n\sin\theta \pm \sin\theta)d\theta \geq 0 \qquad (\mathrm{Im}(f(re^{i\theta})) \sim \mathrm{Im}(re^{i\theta}) \sim \sin\theta) \\
\Rightarrow\ & \mathrm{Im}\left(\int_0^{2\pi} f(re^{i\theta})(n\sin\theta \pm \sin n\theta)d\theta\right) \geq 0 \\
\Rightarrow\ & \mathrm{Im}(i\pi n a_1 r \pm i\pi a_n r^n) \geq 0 \\
\Rightarrow\ & \pi n a_1 r \pm \pi a_n r^n \geq 0 \qquad (a_n \in \mathbb{R},\ \forall n \geq 1,\ r > 0) \\
\Rightarrow\ & |a_n| \leq \frac{n|a_1|}{r^{n-1}} \qquad (\forall n \geq 1,\ r > 0)
\end{aligned}$$

Since r can be selected from our choice, picking $r \to \infty$, the above inequality implies $a_k = 0$ for $k \geq 2$. Therefore $f(z) = a_0 + a_1 z$. $\square$

6. **Find by Yourself:** Computation exercise regarding expansion of Laurent series.