

Lecture 20

28/04/2020

- isolated singularity z_0



$\exists \epsilon > 0$ s.t.

f is holomorphic on $\underbrace{B_\epsilon(z_0) \setminus \{z_0\}}_{A_{\epsilon,0}(z_0)}$.

$\Rightarrow f$ can be written as a Laurent's series on $A_{\epsilon,0}(z_0)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$



- removable singularity: $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

- pole of order k : $f(z) = \frac{a_{-k} \neq 0}{(z-z_0)^k} + \frac{a_{(k-1)}}{(z-z_0)^{k-1}} + \dots + a_0$

- essentially singularity: $f(z) = \dots + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

$$f(z) = \dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + a_0 + a_1(z-z_0) + \dots$$

$\exists$ non-zero a_{-n} 's.

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \dots$$

(0 is an essential singularity.)

f has an isolated singularity at z_0 .

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \text{ in } A_{\varepsilon,0}(z_0)$$



$$\oint_{\gamma} f(z) dz = 2\pi i a_{-1}$$

$\uparrow$
 $\text{Res}(f, z_0)$



e.g. $f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$

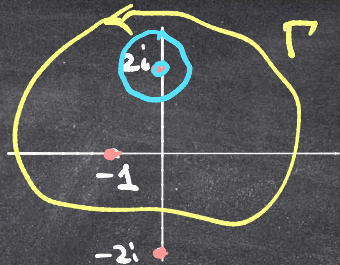
partial fractions $\rightarrow = \frac{\frac{3}{5}}{(z+1)^2} + \frac{\frac{-14}{25}}{z+1} + \frac{\frac{7+i}{25}}{z-2i}$

holds
on $B_\epsilon(2i)$

$$\text{Res}(f, 2i) = \frac{7+i}{25}$$

$$\text{Res}(f, -2i) = \frac{7-i}{25}$$

$$\text{Res}(f, -1) = -\frac{14}{25}$$



$$= a_0 + a_1(z-2i) + a_2(z-2i)^2 + \dots$$

$$\oint_{\Gamma} f(z) dz$$

$$= 2\pi i \left(\text{Res}(f, -1) + \text{Res}(f, 2i) \right)$$

$$= 2\pi i \left(-\frac{14}{25} + \frac{7+i}{25} \right)$$

Find order of pole:

$$f(z) = \frac{c_{-k} \neq 0}{(z-z_0)^k} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \dots + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$$

$$(z-z_0)^m f(z) = c_{-k} (z-z_0)^{m-k} + c_{-(k-1)} (z-z_0)^{m-k+1} + \dots$$

$$\lim_{z \rightarrow z_0} (z-z_0)^m f(z) = \begin{cases} 0 & \text{if } m > k \\ c_{-k} & \text{if } m = k \\ \text{doesn't exist} & \text{if } m < k \end{cases}$$

To determine the order of pole z_0 ,

find $m \in \mathbb{N}$ s.t. $\lim_{z \rightarrow z_0} (z-z_0)^m f(z)$ exists
and non-zero

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

find

order of -1 ?

$$\lim_{z \rightarrow -1} (z+1)^{\textcircled{1}} f(z) \text{ exist?}$$

$$\lim_{z \rightarrow -1} (z+1)^4 f(z) = \lim_{z \rightarrow -1} \frac{(z+1)^2 (z^2 - 2z)}{(z^2 + 4)} = 0$$

$$\lim_{z \rightarrow -1} (z+1)^1 f(z) = \lim_{z \rightarrow -1} \frac{z^2 - 2z}{(z+1)(z^2 + 4)}$$

does not exist.

$\downarrow 0$ $\rightarrow \frac{3}{5}$

$$\lim_{z \rightarrow -1} (z+1)^{\textcircled{2}} f(z) = \lim_{z \rightarrow -1} \frac{z^2 - 2z}{z^2 + 4} = \frac{3}{5} \neq 0.$$

$\hookrightarrow \infty$.

$\therefore$ pole of order 2.

$$f(z) = \frac{c_{-k} \leftarrow \neq 0}{(z-z_0)^k} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \dots + \boxed{c_{-1}} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$$

$\boxed{c_{-1}} = ?$

pole of order k .

$$(z-z_0)^k f(z) = c_{-k} + c_{-(k-1)}(z-z_0) + \dots + \boxed{c_{-1}}(z-z_0)^{k-1} + c_0(z-z_0)^k + \dots$$

$\leftarrow$

$(z-z_0)^{k-1} \rightarrow (k-1)(z-z_0)^{k-2} \rightarrow (k-1)(k-2)\frac{z^{k-3}}{z}$

$$\frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) = \boxed{c_{-1}} (k-1)! + b_0(z-z_0) + b_1(z-z_0)^2 + \dots$$

$\boxed{c_{-1}} = ?$

$$\lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) = c_{-1} \cdot (k-1)! + 0 + \dots$$

$$\| \text{Res}(f, z_0) = c_{-1} = \frac{1}{(k-1)!} \lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) \|$$

$$\text{e.g. } f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

$$\text{ord}_f(-1) = 2$$

$$\text{Res}(f, -1) = \frac{1}{1!} \lim_{z \rightarrow -1} \frac{d}{dz} (z+1)^2 f(z)$$

$$k=2$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4} \right)$$

$$= \lim_{z \rightarrow -1} \frac{(z^2 + 4)(2z - 2) - (z^2 - 2z) \cdot 2z}{(z^2 + 4)^2}$$

$$= \frac{(1+4)(-4) - (1+3) \cdot (-2)}{5^2}$$

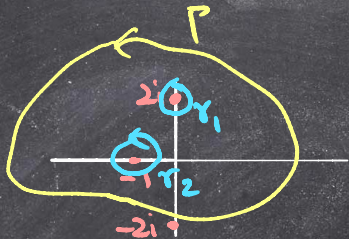
$$= \frac{-14}{25}$$

$$\text{Res}\left(\frac{1}{z}, 0\right) = 1$$

↑

$$+0+\dots+0+\frac{1}{z}+0+0+\dots+$$

↑



$$\oint_{\Gamma} f(z) dz = \oint_{r_1} f(z) dz + \oint_{r_2} f(z) dz.$$

$$= 2\pi i \text{Res}(f, -1) + 2\pi i \text{Res}(f, 2i)$$

$$\oint f(z) dz = 2\pi i \text{Res}(f, -1)$$

↻

$$= 2\pi i \left(-\frac{14}{25}\right).$$

$$\bullet \quad f(z) = \frac{\pi}{z^2} \cot \pi z = \frac{\pi \cos \pi z}{z^2 \sin \pi z}$$

$$\text{Res}(f, 0) = ?$$

Pole order 3 : $\lim_{z \rightarrow 0} z^3 f(z) = \lim_{z \rightarrow 0} \frac{\pi z \cos \pi z}{\sin \pi z} = 1.$

(Note: In the original image, the numerator $\pi z \cos \pi z$ is circled in blue with an arrow pointing to a blue '1', and the denominator $\sin \pi z$ is circled in yellow with an arrow pointing to a yellow '1'.)

$$\text{Res}(f, 0) = \underset{\substack{\uparrow \\ k=3}}{\frac{1}{2!}} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} z^3 f(z)$$

$$= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \frac{\pi z \cos \pi z}{\sin \pi z}$$

$$= \frac{1}{2} \lim_{z \rightarrow 0} \frac{\pi^2 (\pi z \cos \pi z - \sin \pi z)}{\sin^3 \pi z}$$

$$= \frac{1}{2} \lim_{z \rightarrow 0} \frac{\pi^2 \left(\cancel{\pi z} \left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} + \dots \right) - \left(\cancel{\pi z} - \frac{(\pi z)^3}{3!} + \dots \right) \right)}{(\sin \pi z)^3}$$

$$= \frac{\pi^2}{2} \lim_{z \rightarrow 0} \frac{\left(-\frac{1}{2!} + \frac{1}{3!} \right) (\pi z)^3 + \dots}{(\sin \pi z)^3}$$

(Note: In the original image, a blue circle highlights the term $(\pi z)^3$ in the numerator and $(\sin \pi z)^3$ in the denominator. A blue arrow points from the exponent 3 in the denominator to the term $(\pi z)^3$ in the numerator. A yellow circle highlights the term $(\pi z)^5$ in the numerator, with a blue arrow pointing to it from the expression $3+2$ written above it. A yellow arrow points from the ellipsis in the numerator to the ellipsis in the denominator.)

$$= \frac{\pi^2}{2} \cdot \left(-\frac{1}{2} + \frac{1}{6} \right)$$