Lecture 20
28/04/2020

- isolated singularity z_0

$\exists \varepsilon > 0$ s.t. f is holomorphic on $\underbrace{B_\varepsilon(z_0) \setminus \{z_0\}}_{A_{\varepsilon,0}(z_0)}$.

$\Rightarrow f$ can be written as a Laurent's series on $A_{\varepsilon,0}(z_0)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

- removable singularity: $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

- pole of order k: $f(z) = \dfrac{\overset{\neq 0}{a_{-k}}}{(z-z_0)^k} + \dfrac{a_{-(k-1)}}{(z-z_0)^{k-1}} + \dots + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

- essentially singularity:

$$f(z) = \dots + \frac{a_{-n}}{(z-z_0)^n} + \dots + a_0 + a_1(z-z_0) + \dots$$

$\exists\, \infty$ non-zero a_{-n}'s.

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \cdots$$

(0 is an entially singularity.)

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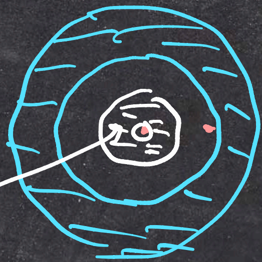
f has an isolated singularity at z_0.

$\Rightarrow \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ in $A_{\varepsilon,0}(z_0)$

$$\oint_{\gamma} f(z)\,dz = 2\pi i \, \underbrace{a_{-1}}_{\uparrow\ \mathrm{Res}(f,\, z_0).}$$



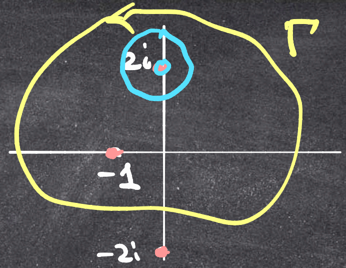
e.g. $\quad f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

$$= \frac{\frac{3}{5}}{(z+1)^2} + \frac{-\frac{14}{25}}{z+1} + \frac{\frac{7+i}{25}}{z-2i} + \frac{\frac{7-i}{25}}{z+2i}$$

partial fractions

holo. on $B_\varepsilon(2i)$

$$= a_0 + a_1(z-2i) + a_2(z-2i)^2 + \cdots$$



$$\mathrm{Res}(f, 2i) = \frac{7+i}{25}$$

$$\mathrm{Res}(f, -2i) = \frac{7-i}{25}$$

$$\mathrm{Res}(f, -1) = -\frac{14}{25}$$

$$\oint_\Gamma f(z)\,dz$$

$$= 2\pi i\left(\mathrm{Res}(f,-1) + \mathrm{Res}(f, 2i)\right)$$

$$= 2\pi i\left(-\frac{14}{25} + \frac{7+i}{25}\right)$$

Find order of pole:

$$f(z) = \overset{\neq 0}{\frac{c_{-k}}{(z-z_0)^k}} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

$$(z-z_0)^m f(z) = c_{-k}(z-z_0)^{m-k} + c_{-(k-1)}(z-z_0)^{m-k+1} + \cdots$$

$$\lim_{z\to z_0} (z-z_0)^m f(z) = \begin{cases} 0 & \text{if } m>k \\ c_{-k} & \text{if } m=k \\ \text{doesn't exist} & \text{if } m<k \end{cases}$$

To determine the order of pole z_0,

find $m \in \mathbb{N}$ s.t. $\lim\limits_{z\to z_0} (z-z_0)^m f(z)$ exists and non-zero

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

find order of -1 ?

$$\lim_{z \to -1} (z+1)^{?} f(z) \quad \text{exist?}$$

$$\lim_{z \to -1} (z+1)^4 f(z) = \lim_{z \to -1} \frac{(z+1)^2 (z^2 - 2z)}{(z^2+4)} = 0$$

$$\lim_{z \to -1} (z+1)^1 f(z) = \lim_{z \to -1} \frac{z^2 - 2z}{\underbrace{(z+1)}_{\to 0}(z^2+4)} \quad \text{does not exist.}$$

$$\to \frac{3}{5}$$

$$\lim_{z \to -1} (z+1)^{2} f(z) = \lim_{z \to -1} \frac{z^2 - 2z}{z^2+4} = \frac{3}{5} \neq 0.$$

$$< \infty.$$

$\therefore$ pole of order 2.

$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

$\neq 0$

$= ?$

pole of order k.

$$(z-z_0)^{k-1} \rightsquigarrow (k-1)(z-z_0)^{k-2} \rightsquigarrow (k-1)(k-2)z^{k-3}$$

$$(z-z_0)^k f(z) = c_{-k} + c_{-(k-1)}(z-z_0) + \cdots + c_{-1}(z-z_0)^{k-1} + c_0(z-z_0)^k + \cdots$$

$= ?$

$$\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}(k-1)! + b_0(z-z_0) + b_1(z-z_0)^2 + \cdots$$

$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}\cdot(k-1)! + 0 + \cdots$$

$$\left\| \operatorname{Res}(f, z_0) = c_{-1} = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) \right\|$$

e.g. $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

$\operatorname{ord}_f(-1) = 2$

$$\operatorname{Res}(f, -1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} (z+1)^2 f(z)$$

$k = 2$

$$= \lim_{z \to -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4} \right)$$

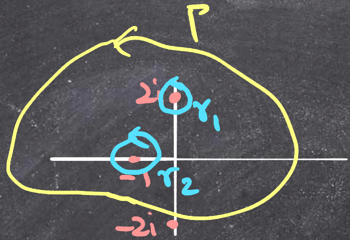
$$= \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z) \cdot 2z}{(z^2+4)^2}$$

$$= \frac{(1+4)(-4) - (1+3) \cdot (-2)}{5^2}$$

$$= \frac{-14}{25}.$$

$$\text{Res}\left(\frac{1}{z}, 0\right) = 1$$

$$+ 0 + \cdots 0 + \frac{1}{z} + 0 + 0 + \cdots +$$



$$\oint_{\gamma} f(z)\,dz = 2\pi i \,\text{Res}(f, -1)$$

$$= 2\pi i \left(-\frac{14}{25}\right).$$

$$\oint_{\Gamma} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz.$$

$$= 2\pi i \,\text{Res}(f, -1) + 2\pi i \,\text{Res}(f, 2i)$$

- $f(z) = \frac{\pi}{z^2}\cot \pi z = \frac{\pi \cos \pi z}{z^2 \sin \pi z}$

$\text{Res}(f, 0) = ?$

Pole order 3 : $\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{\pi z \overbrace{\cos \pi z}^{\to 1}}{\underbrace{\sin \pi z}_{\to 1}} = 1.$

$$\text{Res}(f.0) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 f(z)$$

$\uparrow$
$k=3$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} \frac{\pi z \cos z}{\sin \pi z}$$

$= \dots$

$$= \frac{1}{2} \lim_{z\to 0} \frac{\pi^2 (\pi z \cos \pi z - \sin \pi z)}{\sin^3 \pi z}$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{\pi^2 \left(\pi z \left(\cancel{1} - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} + \cdots \right) - \left(\cancel{\pi z} - \frac{(\pi z)^3}{3!} + \cdots \right) \right)}{(\sin \pi z)^3}$$

3+2

$$= \frac{\pi^2}{2} \lim_{z \to 0} \frac{\left(-\frac{1}{2!} + \frac{1}{3!} \right)(\pi z)^3 + ?(\pi z)^5 + \cdots}{(\sin \pi z)^3}$$

1

0

$$= \frac{\pi^2}{2} \cdot \left(-\frac{1}{2} + \frac{1}{6} \right)$$