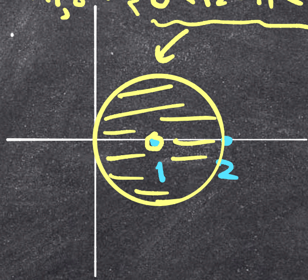
Lecture 19
23/04/2020

• $f(z) = \dfrac{1}{(z-1)(z-2)} : \mathbb{C}\setminus\{1,2\} \to \mathbb{C}$.

$A_{1,0}(1) = \{0<|z-1|<1\}$



$f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n (z-1)^n$?

$f(z) = \underbrace{\dfrac{1}{z-1}}_{\text{Keep.}} \cdot \dfrac{1}{z-2} = \dfrac{1}{z-1}\cdot\dfrac{1}{(z-1)-1}$

$= -\dfrac{1}{z-1}\cdot\dfrac{1}{1-(z-1)}$

$|z-1|<1$

$\dfrac{1}{1-w} = \sum w^n$.

$|w|<1$

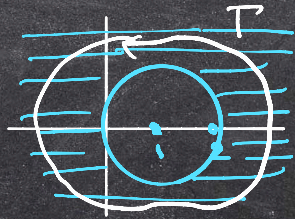
$= -\dfrac{1}{z-1}\displaystyle\sum_{n=0}^{\infty} (z-1)^n$

$= -\displaystyle\sum_{n=0}^{\infty} (z-1)^{n-1} = -\dfrac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots$

$$\oint_\gamma f(z)\,dz = \oint_\gamma -\frac{1}{z-1}\,dz + 0 + 0 + \cdots = -2\pi i$$

$\gamma \subset \underbrace{A_{1,0}(1)}$

On $\underbrace{A_{\infty,1}(1)}$



$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{(z-1)-1}$$

$$= \frac{1}{z-1}\cdot\frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)}$$

$$= \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n} = \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+2}}.$$

$A_{\infty,1}(1)$

$$= \{1 < |z-1|\}$$

$$\underbrace{\left|\boxed{\frac{1}{z-1}}\right|}_{w} < 1$$

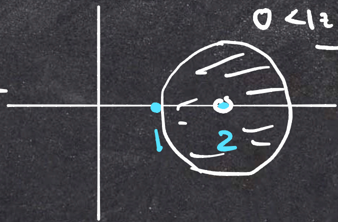
$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} + \cdots$$

$$\oint_{\Gamma} f(z)\,dz = \oint_{\Gamma} \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} + \cdots$$

$\Gamma \subset A_{0,1}(1)$

$$= 0 + 0 + 0 + \cdots = 0.$$

$0 < |z-2| < 1$



$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)+1} \cdot \frac{1}{z-2}$$

$$= \frac{1}{\underbrace{1-(-(z-2))}_{w = -(z-2)}} \cdot \frac{1}{z-2}$$

$$z^2 e^{\frac{1}{z}} = ?$$

$$A_{\infty,0}(0) = \{0 < |z|\}.$$

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

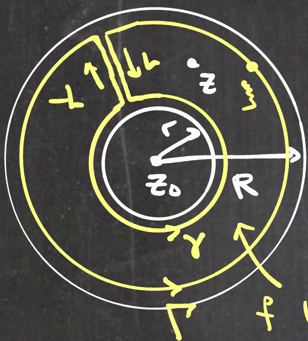
$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}.$$

$$z^2 e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^{n-2}} = \cdots + \underbrace{\frac{1}{3!}\frac{1}{z}}_{n=3} + \cdots$$

$$\oint_{|z|=R} z^2 e^{\frac{1}{z}}\,dz = \oint_{|z|=R} \frac{1}{3!}\frac{1}{z}\,dz = \frac{2\pi i}{6}$$

$$= \frac{\pi i}{3}.$$

Why Laurent's series expansion always exists for a holomorphic function on $\underbrace{A_{R,r}(z_0)}_{\text{annulus}}$?



Cauchy's integral formula

$$\Rightarrow \frac{1}{2\pi i}\underbrace{\left(\oint_\Gamma + \cancel{\int_L} - \oint_\gamma - \cancel{\int_L}\right)}_{\oint}\frac{f(\xi)}{\xi - z}\,d\xi = f(z).$$

$$\Rightarrow \underbrace{\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi-z}\,d\xi}_{(\mathrm{I})} - \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi-z}\,d\xi}_{(\mathrm{II})} = f(z).$$

Similar to Taylor series:

$$(\mathrm{I}) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi-z}\,d\xi$$

$$= \sum_{n=0}^{\infty}\overbrace{\left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi\right)}^{a_n}(z-z_0)^n.$$

$$(\mathrm{II}) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi-z}\,d\xi$$

$$\gamma := \partial B_{r+\varepsilon}(z_0) = \{|z-z_0| = r+\varepsilon\}$$

$\underline{\varepsilon > 0}$.

$$|z-z_0| > r+\varepsilon$$

$$\frac{1}{\xi - z} = \frac{1}{(\xi-z_0)-(z-z_0)}$$

$$= \frac{1}{z-z_0}\cdot\frac{1}{\underbrace{\frac{\xi-z_0}{z-z_0}}_{=:w}-1}$$

$$|\xi - z_0| = r+\varepsilon < |z-z_0| \Rightarrow |w| < 1.$$

$$\frac{1}{\xi - z} = \frac{-1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^n \qquad (\because \left| \frac{\xi - z_0}{z - z_0} \right| < 1).$$

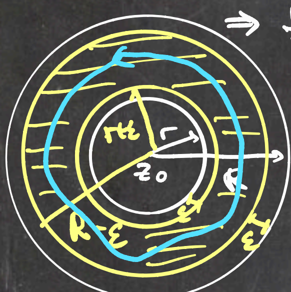
$$\frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} d\xi = -\frac{1}{2\pi i} \oint_\gamma \sum_{n=0}^{\infty} f(\xi) \cdot (\xi - z_0)^n \, d\xi \cdot \frac{1}{(z - z_0)^{n+1}}$$

$$= -\sum_{n=0}^{\infty} \underbrace{\boxed{\frac{1}{2\pi i} \left(\oint_\gamma f(\xi)(\xi - z_0)^n d\xi \right)}}_{a_{-(n+1)}} \cdot \frac{1}{(z - z_0)^{n+1}}$$

$$\Rightarrow f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots} + \underbrace{\sum_{n=0}^{\infty} a_{-(n+1)} \frac{1}{(z - z_0)^{n+1}}}_{\frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots}$$

<u>Fact:</u> f holo. on $A_{R\,r}(z_0)$.

$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ converges uniformly on $\boxed{A_{R-\varepsilon, r+\varepsilon}(z_0)}$ <u>$\forall \varepsilon > 0$</u>.

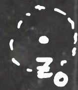


$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \, dz$$

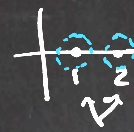
$$= \sum_{n=-\infty}^{\infty} \oint_\gamma a_n (z - z_0)^n \, dz$$

uniform convergence on $\gamma \subset A_{R-\varepsilon, r+\varepsilon}(z_0)$

Isolated singularity.



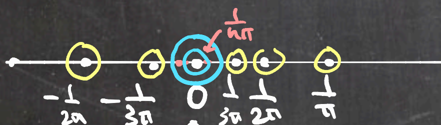
$$f(z) = \frac{1}{(z-1)(z-2)}$$



isolated singularities.

$$h(z) = \frac{1}{\sin \frac{1}{z}}.$$

$$z \neq \frac{1}{n\pi} \quad (n \in \mathbb{Z}) \quad n \neq 0.$$



not isolated.

z_0 : isolated singularity of f ← f is holo. on $B_\varepsilon(z_0) \setminus \{z_0\}$.

- removable singularity $\overset{\text{def}}{\Longleftrightarrow}$ $f(z) = \underbrace{\sum_{n=0}^{\infty}}_{\text{no negative } n.} a_n (z-z_0)^n$

  e.g.

  $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

  $$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

  0 is a removable singularity of $\frac{\sin z}{z}$.

- pole of order k

  $\boxed{\text{pole of order 1} =: \text{simple pole}.}$

$$\overset{\text{def}}{\Longleftrightarrow} f(z) = \frac{\overset{\neq 0}{a_{-k}}}{(z-z_0)^k} + \frac{a_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + a_0 + a_1(z-z_0) + \cdots$$