MATH4023
Tutorial 10 - Taylor Series and Laurent Series
April 28/30, 2020 Notes by Ivan So

1 Review

Theorem 1.1. (Taylor's theorem for holomorphic function) Suppose f is *holomorphic* in an open disc $B_R(z_0)$, then for any $z \in B_R(z_0)$, the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

converges *pointwise* to $f(z)$. This is known as the **Taylor's series** of the holomorphic function f at z_0.

Proof sketch: Let $R' = R - \epsilon$ for some $\epsilon > 0$ such that $z \in B_{R'}(z_0)$. Consider $f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R'} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R'} \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1-\frac{z-z_0}{\zeta-z_0}} d\zeta$.

Corollary 1.2. (Taylor remainder for holomorphic function) Let f satisfying the conditions as in the Taylor's theorem, then for any closed curve $\gamma \subset B_R(z_0)$ enclosing z and z_0

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \underbrace{\frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z}\left(\frac{z-z_0}{\zeta-z_0}\right)^N d\zeta}_{=:R_N(z),\text{ the remainder term}}$$

Proof sketch: Similar to the proof of Taylor's theorem, but notice that $\frac{1-\left(\frac{z-z_0}{\zeta-z_0}\right)^N}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)} = \sum_{n=1}^{N-1} \frac{1}{(z-z_0)^n}$.

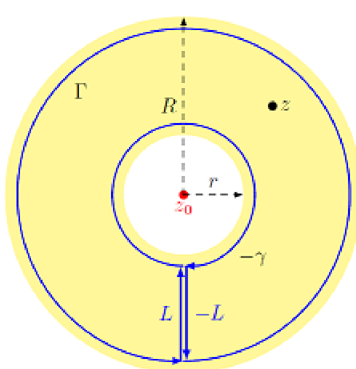
Theorem 1.3. (Laurent) Suppose f is a *holomorphic* function defined on the *annulus* $A_{r,R}(z_0)$ for $r, R \in [0, \infty]$ and $r < R$, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

for some complex numbers c_n. The series is known as the **Laurent series** of the function f at z_0.

Proof sketch:

- Consider the keyhole domain



- In the defined domain we have

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\oint_{\Gamma} \frac{1}{\zeta - z_0}\frac{f(\zeta)}{1-\frac{z-z_0}{\zeta-z_0}} d\zeta + \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{z - z_0}\frac{f(\zeta)}{1-\frac{\zeta-z_0}{z-z_0}} d\zeta.$$

  Expand the series.

Corollary 1.4. (Laurent remainder) Let f be a function meeting the requirement of Laurent's theorem, then for each $N \in \mathbb{Z}^+$ and $z \in A_{R,r}(z_0)$,

$$f(z) = \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_\gamma f(\zeta)(\zeta - z_0)^{n-1} d\zeta \right) \frac{1}{(z-z_0)^n} + \underbrace{\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{z-\zeta} \left(\frac{\zeta - z_0}{z - z_0} \right)^N d\zeta}_{=:r_N(z)}$$
$$+ \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \underbrace{\frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{z-\zeta} \left(\frac{z - z_0}{\zeta - z_0} \right)^N d\zeta}_{=:R_N(z)}$$

where Γ and γ are any pair of circles in $A_{R,r}(z_0)$ centered at z_0 such that z is in between Γ and γ.

Proof sketch: Similar to the proof of Taylor's theorem through considering a geometric sum.

2 Problems

1. **True or False**

   (a) Laurent's theorem is a consequence of Cauchy-Goursat's theorem.

   (b) Suppose $f(z)$ is defined on $B_r(z_0)$ has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then $f(z)$ is holomorphic.

2. Suppose $|f(z)| \leq A + B|z|^k$ and f is entire. Show that all the coefficients c_j, $j > k$ in it's power series expansion are 0.

3. (Weierstrass theorem) Prove that if f is holomorphic on $\mathbb{C} \setminus \{z_1, \cdots, z_N\}$, then there are $N+1$ entire functions $f_0, \cdots, f_N$ such that
$$f(z) = f_0(z) + f_1(1/(z - z_1)) + \cdots + f_N(1/(z - z_N)).$$

4. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an *entire* function? Justify your answer.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, whose Taylor's series about 0 is given by:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

   (a) Show that for any integer $n \geq 1$ and $r \in \mathbb{R}^{>0}$, we have:

$$\int_0^{2\pi} f(re^{i\theta}) \sin n\theta d\theta = i\pi a_n r^n.$$

   (b) Two real numbers α and β are said to have the same sign if they are both positive or both negative or both zero. Suppose further that for each $z \in \mathbb{C}$, $\mathrm{Im}(f(z))$ and $\mathrm{Im}(z)$ always have the same sign.

      (i) Show that $a_n \in \mathbb{R}$ for any $n \geq 0$.

      (ii) Using the result of (i), show that

$$f(z) = a_0 + a_1 z$$

      for any $z \in \mathbb{C}$. [Hint: You can use without proofs that $n \sin\theta + \sin n\theta$, $n \sin\theta - \sin n\theta$ and $\sin\theta$ all must have the same sign for any integer $n \geq 2$ and any $\theta \in [0, 2\pi]$.]

6. **Find by Yourself:** Computation exercise regarding expansion of Laurent series.