Lecture 15
09/04/2020

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f is holomorphic on $\Omega \subset \mathbb{C}$, ← open

then $f^{(n)}$ exists on Ω $\forall n \in \mathbb{N}$.

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§ 3.5 - Morera's Theorem.

Thm. 3.18

Given: $\left\{ \begin{array}{l} f : \underset{\text{open}}{\Omega} \to \mathbb{C} \quad \text{continuous function.} \\ \oint_T f(z)\,dz = 0 \quad \forall \text{ Triangle } T \subset \Omega. \end{array} \right.$

T → any triangle $\subset \Omega$

$\Rightarrow$ f is holomorphic on Ω.

$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\,dt$ $\qquad$ $\underline{\mathrm{Re}(z) > 0}$ $\qquad$ $z = x + yi$

$$|t^{z-1} e^{-t}| = t^{x-1} e^{-t}.$$

$$\int_0^\infty t^{x-1} e^{-t}\,dt < \infty.$$

$$\oint_T \Gamma(z)\,dz = \oint_T \int_0^\infty t^{z-1} e^{-t}\,dt\,dz$$

$\forall t \neq 0,$ holo. in z.

$$\overset{?}{=} \int_0^\infty \underbrace{\oint_T t^{z-1} e^{-t}\,dz}_{=0 \text{ by C.G.}}\,dt$$

$$t^{z-1} := e^{(z-1)\ln t}.$$

$$= 0.$$

<u>e.g.</u> $f : \Omega := \{z : \mathrm{Re}(z) < 0\} \to \mathbb{C}$

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt.$$

$$\left|\frac{e^{zt}}{t+1}\right| = \frac{e^{xt}}{t+1} < e^{xt} \quad (x<0)$$

$$\int_0^\infty e^{xt}\,dt = \frac{1}{x}e^{xt}\Big|_0^\infty = -\frac{1}{x}. < \infty.$$

<u>Need</u>:

(1) $f : \Omega \to \mathbb{C}$ cts on Ω.

(2) $\oint_\Delta f(z)\,dz = 0 \quad \forall \Delta \subset \Omega$.

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(1) Need: $z_0 \in \Omega \rightsquigarrow \lim_{z\to z_0} f(z) = f(z_0)$.

$$\Leftrightarrow \lim_{z\to z_0} \int_0^\infty \frac{e^{zt}}{t+1}\,dt = \int_0^\infty \frac{e^{z_0 t}}{t+1}\,dt.$$

?

<u>WANT</u>: $\forall z_n \to z_0$, we have $\lim_{n\to\infty} \int_0^\infty \underbrace{\frac{e^{z_n t}}{t+1}}_{F_n(t)}\,dt = \int_0^\infty \frac{e^{z_0 t}}{t+1}\,dt$

- $\left|\frac{e^{z_n t}}{t+1}\right| = \frac{e^{x_n t}}{t+1} \underset{\substack{\uparrow \\ n \text{ large}}}{\leq} \underbrace{e^{\frac{1}{2}x_0 t}}_{g(t)}$ $\forall n$ large. $\forall t \geq 0$. $z_n = x_n + i y_n$

$z_0 = x_0 + i y_0$

- $\int_0^\infty g(t)\,dt$

$= \int_0^\infty e^{\frac{1}{2}x_0 t}\,dt = -\frac{2}{x_0} < \infty.$

$\uparrow$

$x_0 < 0$.

$\underline{\text{LDCT}} \Rightarrow \underbrace{\lim_{n\to\infty} \int_0^\infty \frac{e^{z_n t}}{t+1}\,dt} = \int_0^\infty \lim_{n\to\infty} \frac{e^{z_n t}}{t+1}\,dt = \int_0^\infty \frac{e^{z_0 t}}{t+1}\,dt.$

$\therefore \lim_{z\to z_0} f(z) = f(z_0) \quad \forall z_0 \in \Omega.$

$\uparrow$

$\therefore$ cts.

(2): Need to show $\oint_T f(z)\,dz = 0 \quad \forall T < \Omega$.

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt$$

$$\oint_T f(z)\,dz = \oint_T \int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz \overset{?}{=} \int_0^\infty \oint_T \frac{e^{zt}}{t+1}\,dz\,dt$$

Fubini Theorem:

If one of the following is finite:
$$\int_0^\infty \oint_T |G(z,t)\,dz\,dt|$$

$$\oint_T \int_0^\infty |G(z,t)\,dt\,dz|$$

then $$\int_0^\infty \oint_T G(z,t)\,dz\,dt = \oint_T \int_0^\infty G(z,t)\,dt\,dz$$

$$\oint_T \int_0^\infty \left| \frac{e^{zt}}{t+1}\, dt\, dz \right| = \oint_T \int_0^\infty \frac{e^{xt}}{t+1}\, dt\, \underbrace{|dz|}_{\sqrt{(dx)^2+(dy)^2}}$$

$$\le \oint_T \underbrace{\int_0^\infty e^{xt}\, dt}\, |dz|$$

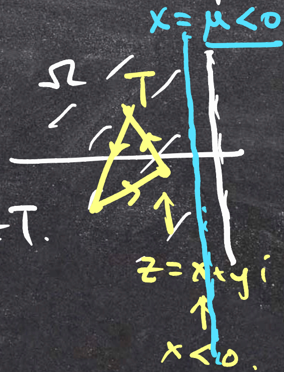
$$= \oint_T -\frac{1}{x}\, |dz| \qquad (x < 0)$$

$x = \mathrm{Re}(z)$, $z \in T$.

$x < \mu < 0$

$-\frac{1}{x} < -\frac{1}{\mu}$.

$$\le \oint_T -\frac{1}{\mu}\, |dz| = -\frac{1}{\mu} L(T) < \infty.$$



Fubini $\Rightarrow$ $$\oint_T \boxed{\int_0^\infty \frac{e^{zt}}{t+1}\,dt}\,dz = \int_0^\infty \oint_T \frac{e^{zt}}{\underbrace{t+1}}\,dz\,dt$$ $= 0$ by CG

$\forall t \geq 0$, e^{zt} is holo. in z.

any $T \subset \Omega$. $$= \int_0^\infty 0\,dt = 0.$$

$\Rightarrow$ $\boxed{\int_0^\infty \frac{e^{zt}}{t+1}\,dt}$ is holo. on Ω.

$f_n \rightrightarrows f$ on Ω $\Longrightarrow$ f holomorphic on Ω

$\uparrow$ holomorphic on Ω $\leftarrow$ open.

$\Downarrow$

$\lim_{n\to\infty} \int = \int \lim_{n\to\infty}$

proper. integral.

<u>Proof</u> :

$T \subset \Omega$

$\uparrow$ triangle contour.

$$\oint_T f(z)\,dz = \oint_T \lim_{n\to\infty} f_n(z)\,dz$$

uniform conv.

$L(T) < \infty$.

$$= \lim_{n\to\infty} \underbrace{\oint_T f_n(z)\,dz}_{=0 \text{ by thm. C-G.}}$$

$= 0$.

Morera $\Longrightarrow$ f is holo. on Ω.

$\square$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \text{ is holo. on } \{\mathrm{Re}(z) > 1\} =: \Omega.$$

<u>since</u> $\sum_{n=1}^{N} \frac{1}{n^z} \underset{\text{as } N\to\infty}{\rightrightarrows} \zeta(z)$ on $\Omega_\varepsilon := \{\underline{\mathrm{Re}(z) > 1+\varepsilon}\}$.

$(\varepsilon > 0)$.

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^{\mathrm{Re}(z)}} < \underbrace{\frac{1}{n^{1+\varepsilon}}}_{M_n}.$$

$$\sum_{n=1}^{\infty} M_n = \sum \frac{1}{n^{1+\varepsilon}} < \infty. \quad (p = 1+\varepsilon > 1)$$

Weierstrass M-test : $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on Ω_ε.

$$\underbrace{\frac{1}{n^z}}_{\substack{\| \\ \frac{1}{e^{z\ln(n)}}}}$$

$\frac{1}{e^{z\ln(n)}}$ holo. on Ω_ε

uniform conv. $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^z}$ holo. on Ω_ε.

letting $\varepsilon \to 0^+$

$\therefore$ $\zeta(z)$ is holo. on Ω.

$$\Omega = \{\mathrm{Re}(z) > 1\}.$$