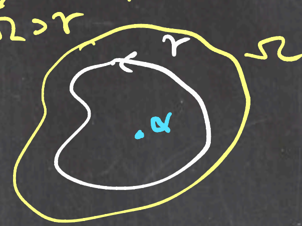
Lecture 13
02/04/2020

holo. on $\Omega \supset \gamma$ (simply-connected)

$$f(\alpha) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{z-\alpha}\,dz$$

γ simple closed



Proof of Cauchy's integral formula:

$$\oint_{\gamma} \frac{f(z)}{z-\alpha}\,dz \overset{\text{Cor. of C-G.}}{=} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz$$

$$= \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)+f(\alpha)}{z-\alpha}\,dz$$

$$= \underbrace{\oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz} + \underbrace{\oint_{|z-\alpha|=\varepsilon} \frac{f(\alpha)}{z-\alpha}\,dz}$$

constant

$$f'(\alpha) = \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha}$$

exists

Take $\varepsilon'=1$, $\exists \delta>0$ s.t. $0<|z-\alpha|<\delta$

$$\Rightarrow \left|\frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha)\right| < \varepsilon' = 1.$$

$$\Rightarrow \left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| - |f'(\alpha)| \le \underbrace{\left|\frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha)\right|}_{<1}$$

$$\Rightarrow \left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| < 1 + |f'(\alpha)|.$$

$(\forall z \in B_\delta(\alpha),\ z \neq \alpha.)$

$$= f(\alpha)\oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz = 2\pi i f(\alpha)$$

$z(t) = \alpha + \varepsilon e^{it}$, $t \in [0, 2\pi]$

$$\oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz = \int_0^{2\pi} \frac{\cancel{i\varepsilon e^{it}}\,dt}{(\cancel{\alpha}+\cancel{\varepsilon e^{it}})-\cancel{\alpha}} = 2\pi i$$

$dz = i\varepsilon e^{it}\,dt$

Whenever $\varepsilon < \delta$, $\left|\dfrac{f(z)-f(\alpha)}{z-\alpha}\right| < 1+|f'(\alpha)|$

$\forall z \in \partial B_\varepsilon(\alpha)$

$$\Rightarrow \left|\oint_{\partial B_\varepsilon(\alpha)} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz\right| < \underbrace{(1+|f'(\alpha)|)}_{M}\cdot\underbrace{2\pi\varepsilon}_{L}$$

$$\lim_{\varepsilon\to 0^+} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = 0.$$

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz + 2\pi i\, f(\alpha)$$

$$\Big\downarrow \ \varepsilon\to 0^+$$

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = 0 + 2\pi i\, f(\alpha)$$

$\blacksquare$

$$f(\alpha) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-\alpha}\, dz$$

$$\left.\begin{matrix} z \to w \\ \alpha \to z \end{matrix}\right\}$$

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z}\, dw$$

Informally:

$$f'(z) = \frac{1}{2\pi i} \frac{d}{dz} \oint_\gamma \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{d}{dz} \frac{f(w)}{w-z}\, dw \qquad \text{(cheating)}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^2}\, dw.$$

$$f''(z) = \frac{1}{2\pi i} \frac{d}{dz} \oint_\gamma \frac{f(w)}{(w-z)^2} dw = \frac{1}{2\pi i} \oint_\gamma \frac{2 \cdot f(w)}{(w-z)^3}\, dw.$$

Inductively:

$$f^{(n)}(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{n!\, f(w)}{(w-z)^{n+1}}\, dw .$$

$$f^{(n)}(\alpha) = \frac{1}{2\pi i} \oint_{\gamma} \frac{n!\, f(z)}{(z-\alpha)^{n+1}}\, dz .$$

Higher-order Cauchy's integral formula.

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e.g.

- $\displaystyle \oint_{|z|=1} \frac{1}{z^2}\, dz = \frac{2\pi i \cdot \underbrace{f'(0)}_{=0}}{1!} = 0 . \quad \begin{cases} \alpha = 0 \\ n = 1 \end{cases}, \quad \frac{f(z)=1}{f'=0} .$

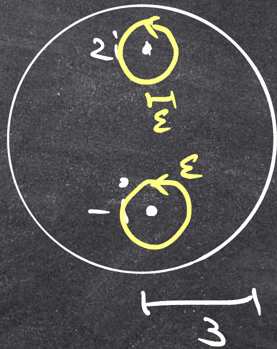
- $\displaystyle \oint_{|z|=1} \frac{e^z}{z^3}\, dz = 2\pi i \cdot \frac{f^{(2)}(0)}{2!} = 2\pi i \cdot \frac{e^z\big|_{z=0}}{2} = \underline{\underline{\pi i}} .$

$\begin{cases} \alpha = 0 \\ n = 2 \end{cases} \quad f(z) = e^z \qquad f'' = e^z$

<u>e.g.</u> $$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\,dz$$

$f_1(z)$ $\qquad\qquad\qquad\qquad$ $f_2(z)$

$$= \oint_{|z-2i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\,dz + \oint_{|z+i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\,dz$$



= . . . .

$$f_1(z) = \frac{1}{(z+i)^2}$$

$$f_1'(z) = -\frac{2}{(z+i)^3}$$

$\vdots$

- $C_0^n - C_1^n + C_2^n - \cdots + (-1)^n C_n^n = 0$

Claim: $\displaystyle \frac{1}{2\pi i} \oint_{|z-1|=r} \frac{z^n}{(z-1)^{k+1}}\, dz = C_k^n \quad \forall r > 0.$

Proof: $\displaystyle \frac{1}{2\pi i} \oint_{|z-1|=r} \frac{\overbrace{z^n}^{=f}}{(\underset{\alpha}{z-1})^{k+1}}\, dz$ $(k \leq n).$

?

Higher-order Cauchy's integral

$$= \frac{f^{(k)}(1)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= C_k^n. \quad \blacksquare$$

$(z^n)' = n z^{n-1}$

$(z^n)'' = n(n-1) z^{n-2}$

$\vdots$

$(z^n)^{(k)} = n(n-1)\cdots(n-(k-1))z^{\cdots}$

$$\sum_{k=0}^{n} (-1)^k C_k^n$$

$$= \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{2\pi i} \oint_{|z-1|=r} \frac{z^n}{(z-1)^{k+1}}\, dz$$

$$= \frac{1}{2\pi i} \oint_{|z-1|=r} z^n \cdot \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{(z-1)^{k+1}}}_{\text{geometric series.}}\, dz$$

common ratio: $-\dfrac{1}{z-1}$.

$$= \frac{1}{2\pi i} \oint_{|z-1|=r} \frac{z^{\cancel{n}\; n-1} \cdot \cancel{\frac{1}{z-1}} \left(1 - \left(-\frac{1}{z-1}\right)^{n+1}\right)}{1 + \frac{1}{z-1}}\, dz$$

$$1 + \frac{1}{z-1} = \frac{\cancel{z}}{\cancel{z-1}}$$

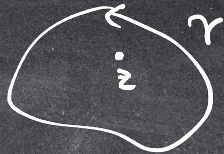
$$= \frac{1}{2\pi i}\oint_{|z-1|=r} z^{n-1}\left(1+(-1)^{n+1}\cdot\frac{1}{(z-1)^{n+1}}\right)dz$$

$$= \frac{1}{2\pi i}\overset{0}{\cancel{\oint_{|z-1|=r} z^{n-1}\,dz}} + \underbrace{\frac{1}{2\pi i}(-1)^{n+1}\oint_{|z-1|=r}\frac{\overset{f(z)}{z^{n-1}}}{(z-\underset{\alpha}{1})^{\underset{n}{n+1}}}\,dz}_{(-1)^{n+1}\cdot\overset{0}{\cancel{\frac{f^{(n)}(1)}{n!}}}}$$

$$= 0$$

∎

- $f'(z) = \dfrac{1}{2\pi i} \oint_{\gamma} \dfrac{f(w)}{(w-z)^2}\, dw$



Liouville's Theorem

$|f| \le M$.

If $f(z)$ is bounded and entire

then $f \equiv$ constant on $\mathbb{C}$.

Proof:
$$\left| \oint_{|w|=R} \frac{f(w)}{(w-z)^2}\, dw \right| \le \underbrace{\frac{M}{(R-|z|)^2} \cdot 2\pi R}_{\approx \frac{1}{R}} \to 0 \text{ as } R \to \infty.$$

On $|w| = R$, $\left| \dfrac{f(w)}{(w-z)^2} \right| \le \dfrac{M}{(R-|z|)^2}$

$$|w-z| \ge |w| - |z|$$

$$\lim_{R\to\infty} \underbrace{\oint_{|w|=R} \frac{f(w)}{(w-z)^2}\,dw}_{2\pi i f'(z)} = 0.$$

R $\to \infty$.

$$\lim_{R\to\infty} \underline{f'(z)} = 0 \implies \left. \begin{array}{l} f'(z) = 0. \\ \forall z \in \mathbb{C}. \end{array} \right\}$$

$$\implies f(z) = \text{constant}.$$

$\square$