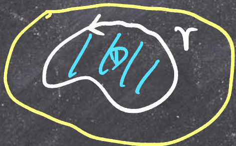
Lecture 12

31/03/2020

Last time:

- Finished the proof of Cauchy-Goursat's Theorem.



$$\oint_\gamma f(z)\,dz = 0.$$

f holomorphic on Ω

$f = u+iv$, $\oint_\gamma f\,\underbrace{dz}_{dx+idy} = \oint_\gamma (u+iv)(dx+idy)$

$= \oint_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$

Green

Need u_x, u_y, v_x, v_y cts $= \iint_D \underbrace{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{=0}$.

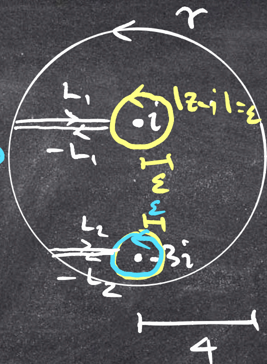
- Cauchy's integral formula:

hol. on Ω

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = 2\pi i\, f(\alpha).$$

$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz \quad = f_1(z)$$

$$+ \oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz \quad = f_2(z)$$



$$= 2\pi i \cdot f_1(i) + 2\pi i \cdot f_2(-3i)$$

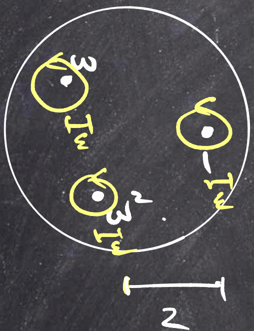
$$= 2\pi i \cdot \frac{i}{i+3i} + 2\pi i \cdot \frac{-3i}{-3i-i} = \frac{\quad}{\quad}$$

e.g. $$\oint_{|z|=2} \frac{1}{z^3-1}\,dz = \oint_{|z|=2} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz .$$

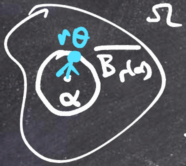
$$= \oint_{|z-1|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz$$

$$+ \oint_{|z-\omega|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz$$

$$+ \oint_{|z-\omega^2|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz$$



Lemma 1: $f: \Omega \to \mathbb{C}$ holomorphic



then:

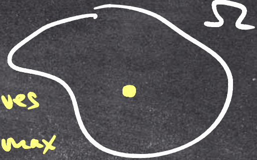
$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\underbrace{\alpha + re^{i\theta}}_{\in \partial B_r(\alpha)})\, d\theta$$

Max. Principle (in PDE)

If u achieves an interior max in Ω,

then

$\underline{u \equiv \text{const}}$.



$\underset{\sim}{\Delta u} = 0$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Proof:

$$f(\alpha) = \frac{1}{2\pi i}\oint_{|z-\alpha|=r} \frac{f(z)}{z-\alpha}\, dz$$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(\alpha + re^{i\theta})}{(\cancel{\alpha} + \cancel{re^{i\theta}}) - \cancel{\alpha}} \cdot \cancel{r}\cancel{i}\cancel{e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta$$

∎

$z(\theta) = \underline{\alpha + re^{i\theta}}$.

$0 \leq \theta \leq 2\pi$

$$dz = \frac{\partial(re^{i\theta})}{\partial\theta}\, d\theta$$

$$= rie^{i\theta}\, d\theta$$

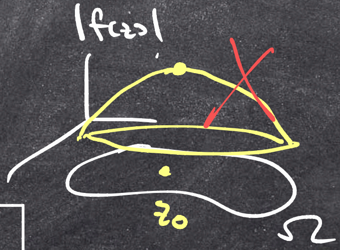
Max. Modulus Theorem

$f : \Omega \to \mathbb{C}$ holo.

open connected

Given $|f(z)|$ achieves max. at $z_0 \in \Omega$.

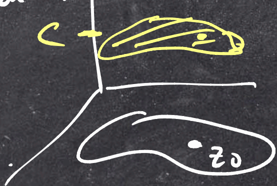
$\Rightarrow f = \text{const.}$ on Ω.



Lemma 2

f is holomorphic on $\overline{B_r(\alpha)}$ and $|f(z)|$ achieves max. on $\overline{B_r(\alpha)}$ at $\underline{\underline{\alpha}}$.

then $f = \text{constant}$ on $\overline{B_r(\alpha)}$.



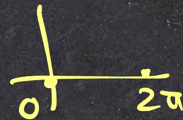
Proof of Lemma 2 :

$$|f(\alpha)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\alpha+re^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(\alpha+re^{i\theta})|}_{\le |f(\alpha)|}\,d\theta$$

$$\le \frac{1}{2\pi}\cdot 2\pi\,|f(\alpha)| = |f(\alpha)|$$

$$\Rightarrow \quad |f(\alpha)| = \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha+re^{i\theta})|\,d\theta .$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\underset{\max}{|f(\alpha)|} - |f(\alpha+re^{i\theta})|}_{\ge 0.}\,d\theta = 0.$$

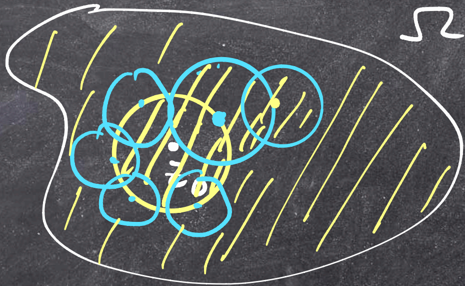


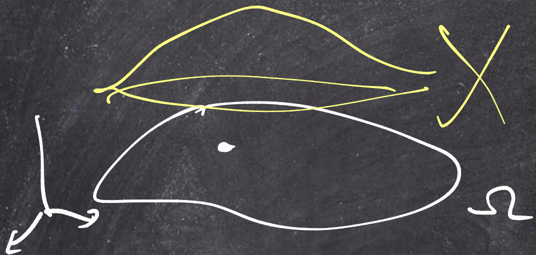
$$|f(\alpha)| = |f(\alpha+re^{i\theta})| \quad \forall \theta \in [0, 2\pi].$$

$$\Rightarrow \quad |f(z)| = |f(\alpha)| \quad \forall z \in \overline{B_r(\alpha)}.$$

Proof of Max. Modulus Theorem:

By Picture:





Fundamental Theorem of Algebra. Proof #1:

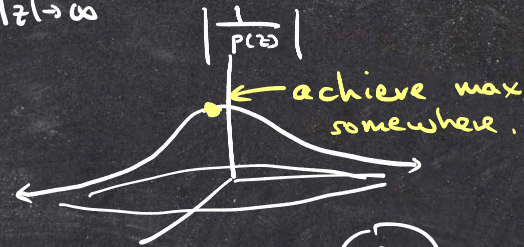
$p(z)$: non-constant complex polynomial.

<u>WANT</u>: $p(z)$ has at least one complex root.

Assume <u>not</u>, then $p(z) \neq 0 \quad \forall z \in \mathbb{C}$

$\Rightarrow \dfrac{1}{p(z)}$ is holomorphic on $\mathbb{C}$ (entire)

$p(z) = a_n z^n + \cdots + a_0 \rightsquigarrow \lim\limits_{|z|\to\infty} |p(z)| = \infty$



$\therefore \lim\limits_{|z|\to\infty} \left|\dfrac{1}{p(z)}\right| = 0$.

Max. Modulus

$\dfrac{1}{p(z)} = \text{const.} \Rightarrow p(z) = \text{const.}$