Lecture 10

24/03/2020

- Cauchy-Goursat's Theorem:

$f : \Omega \to \mathbb{C}$ holomorphic, $\gamma \subset \Omega \implies \oint_\gamma f(z)\,dz = 0.$

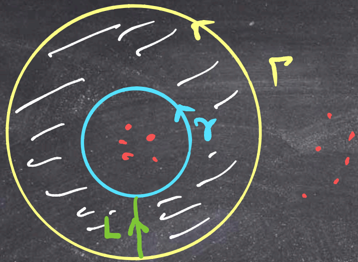
simply-connected

simple closed

- Corollary:

If f is holo.

, then $\oint_\Gamma f(z)\,dz = \oint_\gamma f(z)\,dz.$



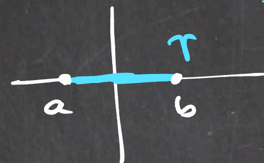
$$-\cancel{\int_L f} + \oint_\Gamma f + \cancel{\int_L f} - \oint_\gamma f$$

$$= \oint f = 0.$$

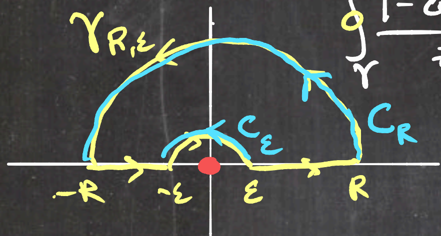
$$\Rightarrow \quad \oint_\Gamma f \, dz = \oint_\gamma f \, dz.$$

e.g. $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = ?$

$\displaystyle\int_{\gamma \subset x\text{-axis}} f(z)\,dz = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx.$



$\displaystyle\oint_\gamma \frac{1-\cos z}{z^2}\,dz = 0$



$\displaystyle\int_{[\varepsilon,R]} \frac{1-\cos z}{z^2}\,dz = \int_\varepsilon^R \frac{1-\cos x}{x^2}\,dx.$

$z(t) = t + 0i$

$a \le t \le b.$

$dz = dt.$

Solution: $\displaystyle f(z) = \frac{1-e^{iz}}{z^2}$ $\qquad$ $\displaystyle f(x+0i) = \frac{1-e^{ix}}{x^2} = \frac{1-\cos x}{x^2} - i\frac{\sin x}{x^2}.$

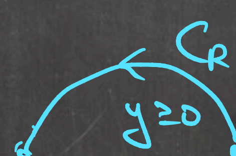
$\displaystyle\int_{[\varepsilon,R]} f(z)\,dz = \int_\varepsilon^R \frac{1-\cos x}{x^2}\,dx - i\int_\varepsilon^R \frac{\sin x}{x^2}\,dx$

$\displaystyle\oint_{\gamma_{R,\varepsilon}} f(z)\,dz = 0$ Cauchy-Goursat.

$\displaystyle\Rightarrow \int_{C_R} f + \int_{-R}^{-\varepsilon} f(x)\,dx - \int_{C_\varepsilon} f + \int_\varepsilon^R f(x)\,dx = 0.$

$\Big\downarrow$ $R \to +\infty$, $\varepsilon \to 0^+$ $\qquad$?

On C_R : $|z| = R$, $\operatorname{Im} z \ge 0$.



$z = x + yi$

$\displaystyle|f(z)| = \left|\frac{1-e^{iz}}{z^2}\right| \le \frac{1+|e^{iz}|}{R^2}$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = |e^{-y}|\underbrace{|e^{ix}|}_{=1}$$

$$= e^{-y} \leq 1.$$

$$(*)$$

$$\therefore \ |f(z)| \leq \frac{2}{R^2} \Longrightarrow \left|\int_{C_R} f(z)\,dz\right| \leq \frac{2}{R^2}\cdot \pi R$$

$$= \frac{2\pi}{R} \to 0$$

as $R \to +\infty$.

On C_ε

$$\int_{C_\varepsilon} f(z)\,dz = \int_{C_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz$$

$$\frac{1-e^{iz}}{z^2} = \frac{\cancel{1} - \left(\cancel{1} + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots + \frac{(iz)^n}{n!} + \cdots\right)}{z^2}$$

$$= -\frac{i}{z} - \frac{i^2}{2!} - \left(\frac{i^3 z}{3!} + \cdots + \frac{i^n z^{n-2}}{n!} + \cdots\right)$$

$$\int_{C_\varepsilon} \frac{1-e^{iz}}{z^2} = \underbrace{-i\int_{C_\varepsilon} \frac{1}{z}\,dz}_{\displaystyle = -i\int_0^\pi \frac{1}{\varepsilon e^{it}}\cdot \varepsilon i e^{it}\,dt = \pi} + \underbrace{\frac{1}{2}\int_{C_\varepsilon} 1\,dz}_{\displaystyle = \frac{1}{2}\cdot \pi\varepsilon} - \int_{C_\varepsilon}\underline{\left(\frac{i^3 z}{3!} + \cdots + \frac{i^n z^{n-2}}{n!} + \cdots\right)}\,dz$$

C_ε

$$z = \varepsilon e^{it}$$

$$\left|\frac{i^3 z}{3!}+\frac{i^4 z^2}{4!}+\cdots+\frac{i^n z^{n-2}}{n!}+\cdots\right|$$ on C_ε

$|z|=\varepsilon$.

$$\le \frac{|z|}{3!}+\frac{|z|^2}{4!}+\cdots+\frac{|z|^{n-2}}{n!}+\cdots$$

$$=\frac{\varepsilon}{3!}+\frac{\varepsilon^2}{4!}+\cdots+\frac{\varepsilon^{n-2}}{n!}+\cdots$$

$$<\varepsilon+\frac{\varepsilon^2}{2!}+\frac{\varepsilon^3}{3!}+\cdots=\underbrace{e^\varepsilon-1}.$$

$$\left|\int_{C_3}\frac{i^3 z}{3!}+\frac{i^4 z^2}{4!}+\cdots dz\right|\le(e^\varepsilon-1)\cdot\pi\varepsilon\to 0$$

as $\varepsilon\to 0^+$.

$$\therefore \lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\frac{1-e^{iz}}{z^2}dz=\pi$$

$$\int_{C_R} f+\int_{-R}^{-\varepsilon} f(x)\,dx-\int_{C_\varepsilon} f+\int_{\varepsilon}^{R} f(x)\,dx=0.$$

$R\to+\infty$, $\varepsilon\to 0^+$ $\downarrow$

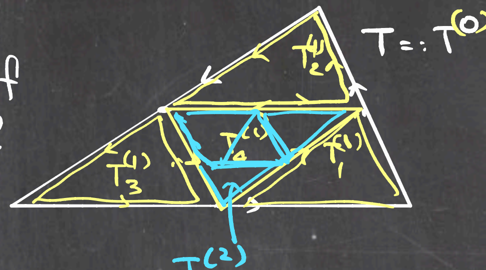
$$0+\int_{-\infty}^{0} f(x)\,dx-\pi+\int_{0}^{\infty} f(x)\,dx=0.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx=\pi \Rightarrow \int_{-\infty}^{\infty}\frac{1-\cos x}{x^2}+\cancel{i\frac{\sin x}{x^2}}\,dx=\pi$$

Proof of Cauchy-Goursat

Step 1: Contour = △ (Special case)

$$\oint_{T^{(0)}} f(z)\,dz = \oint_{T_1^{(1)}} f + \oint_{T_2^{(1)}} f + \oint_{T_3^{(1)}} f + \oint_{T_4^{(1)}} f$$



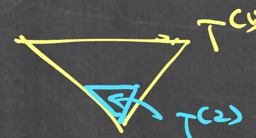
$$\left|\oint_{T^{(0)}} f(z)\,dz\right| \le \sum_{j=1}^{4} \underbrace{\left|\oint_{T_j^{(1)}} f\,dz\right|}$$

$x \le \underbrace{a+b+c+d}.$

$T^{(1)} :=$ → one of these $\geq \frac{1}{4}\left|\oint_{T^{(0)}} f(z)\,dz\right|$

$\max(a,b,c,d) \geq \frac{1}{4}x.$

$$\left|\oint_{T^{(1)}} f\right| \geq \frac{1}{4}\left|\oint_{T^{(0)}} f\right|$$



Repeat : $\exists\, T^{(2)}$ s.t.

$$\left|\oint_{T^{(2)}} f\right| \geq \frac{1}{4}\left|\oint_{T^{(1)}} f\right|$$

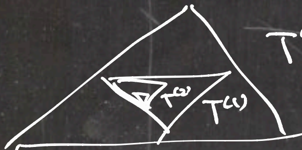
$\therefore \exists\; T^{(1)}, T^{(2)}, T^{(3)}, \ldots$

s.t.
$$\left|\oint_{T^{(n)}} f\right| \geq \frac{1}{4}\underbrace{\left|\oint_{T^{(n-1)}} f\right|} \quad \forall n.$$

$$\geq \frac{1}{4}\cdot\frac{1}{4}\left|\oint_{T^{(n-2)}} f\right|$$

$$\geq \underbrace{\frac{1}{4}\cdots\frac{1}{4}}_{n}\left|\oint_{T^{(0)}} f\right|$$

$$\Rightarrow\; 4^n\left|\oint_{T^{(n)}} f\right| \geq \left|\oint_{T^{(0)}} f\right| \qquad (*).$$



$T^{(0)}$

$\Delta^{(n)} = \overline{\text{interior of } T^{(n)}}$.

$\uparrow$

closed and bounded.

$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \Delta^{(3)} \supset \cdots$

$\exists z_0 \in \bigcap_{n=0}^{\infty} \Delta^{(n)} \subset \Omega$

~~[[[]]]~~

f is complex diff. at z_0.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Rightarrow \lim_{z \to z_0} \frac{\overbrace{f(z) - f(z_0) - f'(z_0)\cdot(z - z_0)}^{=:\, E(z)}}{z - z_0} = 0.$$

$$\oint_{T^{(n)}} f(z) = \oint_{T^{(n)}} \Big(\underbrace{\big(\overbrace{f(z_0) + f'(z_0)(z - z_0)}^{\frac{d}{dz}\left(f(z_0)z + \frac{1}{2}f'(z_0)(z-z_0)^2\right)}\big)}_{} + E(z) \Big)\, dz.$$

$$= 0 + \oint_{T^{(n)}} E(z)\, dz.$$

Progress:

$\forall n \in \mathbb{N}$

$$\oint_{T^{(n)}} f(z)\, dz = \oint_{T^{(n)}} E(z)\, dz.$$